第3章 線形代数

§1 ベクトル

1.1 $\phi(t) = |\vec{a} + t\vec{b}|$

$\{\phi(t)\}^2 = (\vec{a} + t\vec{b})\cdot(\vec{a} + t\vec{b}) = |\vec{a}|^2 + 2t\,\vec{a}\cdot\vec{b} + t^2|\vec{b}|^2$

$= |\vec{b}|^2 \left\{ t + \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2} \right\}^2 + |\vec{a}|^2 - \dfrac{(\vec{a}\cdot\vec{b})^2}{|\vec{b}|^2}$

最小値は $t = \dfrac{-\vec{a}\cdot\vec{b}}{|\vec{b}|^2}$ のとき $\left\{ |\vec{a}|^2 - \dfrac{(\vec{a}\cdot\vec{b})^2}{|\vec{b}|^2} \right\}^{\frac{1}{2}}$

1.2 $\vec{a} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ $\vec{c} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\vec{x} = p\vec{a} + q\vec{b} + r\vec{c}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2p + q + r \\ q - r \\ -p - q + r \end{pmatrix}$

$2p + q + r = x$
$q - r = y$
$-p - q + r = z$

p	q	r	x	y	z
2	1	1	1	0	0
0	1	-1	0	1	0
-1	-1	1	0	0	1
1	0	2	1	0	1
0	1	-1	0	1	0
0	-1	3	1	0	2
1	0	2	1	0	1
0	1	-1	0	1	0
0	0	2	1	1	2
1	0	0	0	-1	-1
0	1	0	$\frac{1}{2}$	$\frac{3}{2}$	1
0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	1

$p = -y - z$

$q = \frac{1}{2}(x + 3y + 2z)$

$r = \frac{1}{2}(x + y + 2z)$

1.3 $\vec{a} = (-1, 2)$ $\vec{b} = (2t, 3)$ $\vec{c} = (1, 3)$

(1) (x_0, y_0) を通る $(x - x_0) + 3(y - y_0) = 0$ $\therefore y = -\frac{1}{3}(x - x_0) + y_0$

(2) $k\vec{a} + 2\vec{b} \perp \vec{c}$

$k\vec{a} + 2\vec{b} = (-k + 4t, 2k + 6)$

$-k + 4t + 6k + 18 = 0$ $4t + 5k = -18$

(3) $|k\vec{a} + 2\vec{b}| = 5$ $(-k + 4t)^2 + (2k + 6)^2 = 25$

(4) $5k^2 - 8kt + 16t^2 + 24k + 11 = 0$ $4t = -18 - 5k$

$5k^2 + 2k(5k + 18) + (5k + 18)^2 + 24k + 11 = 0$, $40k^2 + 240k + 325 = 0$ 解なし

p7

1.4 $|\vec{a}|=|\vec{b}|$ のとき $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2 = 0$

$\therefore \vec{a}+\vec{b} \perp \vec{a}-\vec{b}$

1.5. $\vec{a} = \vec{i}+4\vec{j}$, $\vec{b} = 3\vec{i}+x\vec{j}$ $\quad \vec{a} \perp \vec{b}$ $\quad \vec{a}\cdot\vec{b}=0$

$3+4x=0 \quad x=-\frac{3}{4}$

1.6 $|\vec{a}|=A$, $|\vec{b}|=B$ $\quad \vec{a}$ と $\vec{b}$ のなす角を θ とする

$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

$(\vec{a}\times\vec{b})^2 = A^2B^2\sin^2\theta$, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\therefore (\vec{a}\times\vec{b})^2 + (\vec{a}\cdot\vec{b})^2 = A^2B^2$

§2 一次結合

2.1 $\vec{a}\ \vec{b}\ \vec{c}$ 一次独立ならば $\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=\vec{0}$ は $\alpha=\beta=\gamma=0$ のときに限る.

$x(\vec{a}+\vec{b})+y(\vec{b}+\vec{c})+z(\vec{c}+\vec{a})=\vec{0}$ とすると

$(x+z)\vec{a}+(x+y)\vec{b}+(y+z)\vec{c}=\vec{0}$

$\therefore\ x+z=0 \quad x+y=0 \quad y+z=0$

$\therefore\ x+y+z=0. \quad \therefore\ x=y=0.$

$\therefore\ \vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}$ は一次独立

2.2 $\vec{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$, $\vec{y}=\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$ $\vec{z}=\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}$ が一次従ぞくならば

$\alpha\vec{x}+\beta\vec{y}+\gamma\vec{z}=\vec{0}$ α,β,γ のうち少くとも1つは0でない

$\therefore$ 連立方程式

$$\begin{cases}\alpha x_1+\beta y_1+\gamma z_1=0\\ \alpha x_2+\beta y_2+\gamma z_2=0\\ \alpha x_3+\beta y_3+\gamma z_3=0\end{cases}$$

は $(0,0,0)$ 以外に解をもつ $\therefore$ $\begin{vmatrix}x_1&y_1&z_1\\x_2&y_2&z_2\\x_3&y_3&z_3\end{vmatrix}=0$

逆に $\begin{vmatrix}x_1&y_1&z_1\\x_2&y_2&z_2\\x_3&y_3&z_3\end{vmatrix}=0$ ならば $\vec{x},\vec{y},\vec{z}$ は一次従ぞく

$\therefore$ $\begin{vmatrix}x_1&y_1&z_1\\x_2&y_2&z_2\\x_3&y_3&z_3\end{vmatrix}=0$

2.3 (1) $A=\begin{bmatrix}a&\sqrt{2}b&0&b\\a&0&\sqrt{2}b&c\\a&-\sqrt{2}b&0&b\\a&0&\sqrt{2}c&c\end{bmatrix}$ 基本ベクトル $\vec{e}_1\ \vec{e}_2\ \vec{e}_3\ \vec{e}_4$

$A\vec{e}_1,\ A\vec{e}_2,\ A\vec{e}_3,\ A\vec{e}_4$ が一次独立

$\therefore$ $\begin{vmatrix}a&\sqrt{2}b&0&b\\a&0&\sqrt{2}b&c\\a&-\sqrt{2}b&0&b\\a&0&\sqrt{2}c&c\end{vmatrix}\neq 0$ $\begin{vmatrix}a&\sqrt{2}b&0&b\\0&-\sqrt{2}b&\sqrt{2}b&c-b\\0&-2\sqrt{2}b&0&0\\0&-\sqrt{2}b&\sqrt{2}c&c-b\end{vmatrix}\neq 0$

$2\sqrt{2}ab\begin{vmatrix}\sqrt{2}b&c-b\\\sqrt{2}c&c-b\end{vmatrix}\neq 0$ $4ab(c-b)(b-c)\neq 0$

$a\neq 0 \quad b\neq 0 \quad b\neq c$

(2) $|A\vec{x}|=|\vec{x}|$ $\quad \vec{x}={}^t(x_1\ x_2\ x_3\ x_4)$ とおくと

$|\vec{x}|^2={}^t\vec{x}\cdot\vec{x}$ $\quad |A\vec{x}|^2={}^t\vec{x}\,{}^tA\cdot A\,\vec{x}$ $\quad \therefore {}^t\vec{x}\,\vec{x}={}^t\vec{x}\,{}^tA\cdot A\,\vec{x}$

$$\therefore \begin{pmatrix} a & a & a & a \\ \sqrt{2}b & 0 & -\sqrt{2}b & 0 \\ 0 & \sqrt{2}b & 0 & \sqrt{2}c \\ b & c & b & c \end{pmatrix}\begin{pmatrix} a & \sqrt{2}b & 0 & b \\ a & 0 & \sqrt{2}b & c \\ a & -\sqrt{2}b & 0 & b \\ a & 0 & \sqrt{2}c & c \end{pmatrix}$$

$$= \begin{pmatrix} 4a^2 & 0 & \sqrt{2}a(b+c) & 2a(b+c) \\ 0 & 4b^2 & 0 & 0 \\ \sqrt{2}a(b+c) & 0 & 2(b^2+c^2) & \sqrt{2}c(b+c) \\ 2a(b+c) & 0 & \sqrt{2}c(b+c) & 2(b^2+c^2) \end{pmatrix} = E$$

$\therefore 4a^2=1 \quad 4b^2=1 \quad 2(b^2+c^2)= \quad b+c=0$

$a=\pm\frac{1}{2} \quad b=-c=\pm\frac{1}{\sqrt{2}}$

2.4 $a_1=\begin{bmatrix} \lambda \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$ $\quad a_2=\begin{bmatrix} -\frac{1}{2} \\ \lambda \\ -\frac{1}{2} \end{bmatrix}$ $\quad a_3=\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \lambda \end{bmatrix}$ が一次従属

$\begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix}=0$ $\quad \lambda^3-\frac{3}{4}\lambda-\frac{1}{4}=0$

$$\begin{array}{cccc|c} 4 & 0 & -3 & -1 & 1 \\ & 4 & 4 & 1 & \\ \hline 4 & 4 & 1 & 0 & \end{array}$$

$4\lambda^3-3\lambda-1=0$

$(\lambda-1)(2\lambda+1)^2=0 \quad \lambda=1,\ -\frac{1}{2}$

2.5 $\lambda\neq 1,\ -\frac{1}{2}$

2.6 $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ は

$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix}=0$ のとき一次従属 $\neq 0$ のとき一次独立

$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix}=-1\neq 0$ $\quad \therefore$ 一次独立

2.7 $\vec{a}_1, \vec{a}_2$ が一次独立ならば $\vec{a}_1, \vec{a}_1+\vec{a}_2$ は一次独立

$\because a\vec{a}_1+b(\vec{a}_1+\vec{a}_2)=\vec{0}$, $(a+b)\vec{a}_1+b\vec{a}_2=\vec{0}$ $\quad \therefore a+b=0 \quad b=0$

$\therefore a=b=0$

同様に $\vec{a_1}, \vec{a_2}, \cdots \vec{a_n}$ が一次独立ならば
$\vec{a_1}, \vec{a_1}+\vec{a_2}, \vec{a_1}+\vec{a_2}+\vec{a_3}, \cdots \vec{a_1}+\vec{a_2}+\cdots+\vec{a_n}$ は一次独立

2.8 一次独立であれば4つの列ベクトルからなる行列式は0でない 逆もまた真

$$\begin{vmatrix} 0 & 2 & 1 & 2 \\ -1 & 2 & 5 & 1 \\ 1 & 1 & 3 & 2 \\ -2 & 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 1 & 2 \\ -1 & 0 & 4 & -1 \\ 0 & 1 & 7 & 1 \\ 0 & 1 & -7 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 & 2 \\ 1 & 7 & 1 \\ 1 & -7 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 7 & 1 \\ 0 & 14 & 0 \end{vmatrix}$$

$$= -14\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = 0.$$ ∴ 一次従ぞく

2.9
$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0$$

∴ 一次従ぞく

2.10
$$\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0 \qquad x^3-3x+2=0 \qquad (x-1)^2(x+2)=0$$

$$\begin{array}{rrrr|l} 1 & 0 & -3 & 2 & 1 \\ & 1 & 1 & -2 & \\ \hline 1 & 1 & -2 & 0 & \\ & 1 & 2 & & 1 \\ \hline 1 & 2 & 0 & & \end{array}$$

$x = 1 \quad -2$

2.11 $xz > y^2$

$$\begin{vmatrix} 2x & x+y & x-y \\ 2y & y+z & y-z \\ 3z & z+x & z-x \end{vmatrix} = \begin{vmatrix} 2x & 2x & x-y \\ 2y & 2y & y-z \\ 3z & 2z & z-x \end{vmatrix} = \begin{vmatrix} 0 & 2x & x-y \\ 0 & 2y & y-z \\ z & 2z & z-x \end{vmatrix}$$

$$= 2z\begin{vmatrix} x & x-y \\ y & y-z \end{vmatrix} = 2z(xy - xz - xy + y^2) = 2z(y^2 - xz) \neq 0$$

∴ 一次独立

2.12 $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(1) $\vec{a} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\vec{c} = \begin{pmatrix} 1 \\ 1 \\ x \end{pmatrix}$

$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & x \end{vmatrix} = 0$ $2 - x = 0$ $x = 2$

(2) $\vec{a}, \vec{b}, \vec{c}$ が一次独立なとき $x \neq 2$

$$\begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & x \end{pmatrix} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & x & 0 & 0 & 1 \\ \hline -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1-x & 0 & 1 & -1 \\ 1 & 1 & x & 0 & 0 & 1 \\ \hline -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1-x & 0 & 1 & -1 \\ 0 & 2 & x & 1 & -1 & 1 \\ \hline -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1-x & 0 & 1 & -1 \\ 0 & 0 & 2-x & 1 & 1 & -1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1-x & 0 & 1 & -1 \\ 0 & 0 & 2-x & 1 & 1 & -1 \\ \hline -1 & 0 & 0 & \frac{1}{2-x} & -\frac{1-x}{2-x} & \frac{-1}{2-x} \\ 0 & -1 & 0 & -\frac{1-x}{2-x} & 1-\frac{1-x}{2-x} & -1+\frac{1-x}{2-x} \\ 0 & 0 & 2-x & 1 & 1 & -1 \\ \hline 1 & 0 & 0 & \frac{-1}{2-x} & \frac{1-x}{2-x} & \frac{1}{2-x} \\ 0 & 1 & 0 & \frac{1-x}{2-x} & \frac{-1}{2-x} & \frac{1}{2-x} \\ 0 & 0 & 1 & \frac{1}{2-x} & \frac{1}{2-x} & -\frac{1}{2-x} \end{array}\right)$$

$$\therefore \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix} = \frac{1}{2-x} \begin{pmatrix} -1 & 1-x & 1 \\ 1-x & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix}$$

(3) $\vec{a}, \vec{b}, \vec{c}$ が一次従ぞくのとき

$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & x \end{vmatrix} = 0$ ∴ 逆行列が存在しない

∴ 1次変換も存在しない

2.13 $\begin{vmatrix} a & 2 & a \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{vmatrix} = 0$ $a+3a-2a-2=0$ $a-1=0$

$a=1$

$a=2$ のとき

$\vec{a}=(2,1,2)$ $\vec{b}=(2,1,3)$ $\vec{c}=(2,0,1)$

$\vec{c}-\vec{a}=(0,-1,-1)$ $\vec{b}-\vec{a}=(0,0,1)$

$\vec{p}=\vec{a}+s(\vec{c}-\vec{a})+t(\vec{b}-\vec{a})$ s,t は任意

$=(2,1-s,2-s+t)$

$\vec{p}=(x,y,z)$ とおくと

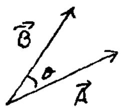
$x=2$, $y=1-s$, $z=2-s+t$

∴ y, z は任意

∴ $x=2$

§3 ベクトルの応用

3.1



$S = |\vec{A}||\vec{B}|\sin\theta$ $\qquad \vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

$\therefore \cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}|\cdot|\vec{B}|} \qquad \sin\theta = \sqrt{1-\cos^2\theta}$

$= |\vec{A}||\vec{B}|\sqrt{1-\dfrac{(\vec{A}\cdot\vec{B})^2}{|\vec{A}|^2|\vec{B}|^2}}$

$= \sqrt{|\vec{A}|^2|\vec{B}|^2-(\vec{A}\cdot\vec{B})^2}$

$\vec{A}\ (ap+b,\ cp+2a^2,\ a^3p-1) \qquad \vec{B}\ (ap-b,\ a^2p+a,\ \frac{p}{a}+a^3)$

$\vec{A}\cdot\vec{B}=0$

$(ap+b)(ap-b)+(cp+2a^2)(a^2p+a)+(a^3p-1)(\frac{p}{a}+a^3)=0$

$a^2p^2-b^2+a^2cp^2+(2a^4+ac)p+2a^3+a^2p^2+(a^6-\frac{1}{a})p-a^3=0$

$(a^2+a^2c+a^2)p^2+(2a^4+ac+a^6-\frac{1}{a})p+a^3-b^2=0$

$a^2(2+c)=0 \qquad 2a^4+ac+a^6-\frac{1}{a}=0 \qquad a^3-b^2=0$

$a\neq 0$ より $c=-2 \qquad a^7+2a^5-2a-1=0 \qquad a^3-b^2=0 \qquad a>0$

1	0	2	0	0	0	−2	−1	1
	1	1	3	3	3	3	1	
1	1	3	3	3	3	1	0	

$\therefore\ a=1,\ b=\pm 1,\ c=-2$

3.2 (1) $\vec{A}=2\vec{i}+2\vec{j}+\vec{k} \qquad |\vec{A}|=\sqrt{4+4+1}=3$

(2) $\vec{e}=\frac{2}{3}\vec{i}+\frac{2}{3}\vec{j}+\frac{1}{3}\vec{k}$

(3) $\vec{B}=2\vec{i}+\vec{j}+4\vec{k}$

$\cos\theta=\dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}=\dfrac{4+2+4}{3\sqrt{21}}=\dfrac{10}{3\sqrt{21}}$

$\theta=\cos^{-1}\dfrac{10}{3\sqrt{21}}$

(4) $\vec{C}=\vec{B}-\vec{A}=-\vec{j}+3\vec{k}$

3.3 $O(0,0) \quad A(x_1,y_1) \quad B(x_2,y_2) \quad C(x_3,y_3)$

(1) $\vec{OA}=(x_1,y_1) \qquad \vec{OB}=(x_2,y_2)$

$\cos\theta=\dfrac{x_1x_2+y_1y_2}{\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}}$

(2) $\triangle OAB=\frac{1}{2}\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)-(x_1x_2+y_1y_2)^2}=\frac{1}{2}\sqrt{(x_1y_2-x_2y_1)^2}$

$=\frac{1}{2}|x_1y_2-x_2y_1|$

(3) $\vec{AB}=(x_2-x_1,\ y_2-y_1) \qquad \vec{AC}=(x_3-x_1,\ y_3-y_1)$

$(x_2-x_1)(y_3-y_1)-(x_3-x_1)(y_2-y_1)=x_2y_3-x_2y_1-x_1y_3+x_1y_1-x_3y_2+x_3y_1+x_1y_2-x_1y_1$

$=x_2y_3+x_3y_1+x_1y_2-x_2y_1-x_1y_3-x_3y_2$

$\therefore\ \triangle ABC=\frac{1}{2}\left|x_2y_3+x_3y_1+x_1y_2-x_2y_1-x_1y_3-x_3y_2\right|$

3.4 $\overrightarrow{P_0P_1}=(x_1-x_0,\ y_1-y_0,\ z_1-z_0)$ $\quad \vec{A}=a\vec{i}+b\vec{j}+c\vec{k}$

(1) $a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)=0$

(2) $\vec{n}_A=\frac{1}{\sqrt{a^2+b^2+c^2}}(a\vec{i}+b\vec{j}+c\vec{k})$

(3) $\overrightarrow{P_0P}=(x-x_0,\ y-y_0,\ z-z_0)$

$\vec{n}_A\cdot\overrightarrow{P_0P}=\frac{1}{\sqrt{a^2+b^2+c^2}}\{a(x-x_0)+b(y-y_0)+c(z-z_0)\}$

(4) $\vec{A}=\vec{i}+\vec{j}+2\vec{k},\quad P_0(2.3.4)$

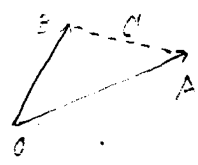
式; $x-2+y-3+2(z-4)=0$

$x+y+2z-13=0$

$\frac{1}{\sqrt{6}}x+\frac{y}{\sqrt{6}}+\frac{2}{\sqrt{6}}z-\frac{13}{\sqrt{6}}=0$

$d=\frac{13}{\sqrt{6}}$ $\quad P_x(13.0.0)\quad P_y(0.13.0)\quad P_z(0.0\ \frac{13}{2})$

3.5 $\overrightarrow{OA}=\vec{a}\quad \overrightarrow{OB}=\vec{b}$



$\overrightarrow{OC}=\overrightarrow{OA}+t(\overrightarrow{OB}-\overrightarrow{OA})$

$=(1-t)\vec{a}+t\vec{b}$ $\qquad 1-t=s$ とおく

$=s\vec{a}+t\vec{b}$ $\qquad t+s=1$

(1) $\frac{\overrightarrow{OA}\cdot\overrightarrow{OC}}{|\overrightarrow{OA}||\overrightarrow{OC}|}=\frac{\overrightarrow{OB}\cdot\overrightarrow{OC}}{|\overrightarrow{OB}|\cdot|\overrightarrow{OC}|}$

$\frac{s|\vec{a}|^2+t\vec{a}\cdot\vec{b}}{|\vec{a}|}=\frac{s\vec{a}\cdot\vec{b}+t|\vec{b}|^2}{|\vec{b}|}$

$s|\vec{a}|\{|\vec{a}|\cdot|\vec{b}|-\vec{a}\cdot\vec{b}\}+t|\vec{b}|\{\vec{a}\cdot\vec{b}-|\vec{a}|\cdot|\vec{b}|\}=0$

$s=1-t$ より

$|\vec{a}|\{|\vec{a}||\vec{b}|-\vec{a}\cdot\vec{b}\}-t(|\vec{b}|+|\vec{a}|)\{|\vec{a}|\cdot|\vec{b}|-\vec{a}\cdot\vec{b}\}=0$

$t=\frac{|\vec{a}|}{|\vec{a}|+|\vec{b}|}\qquad s=\frac{|\vec{b}|}{|\vec{a}|+|\vec{b}|}$

$$\vec{OC} = \frac{|\vec{b}|}{|\vec{a}|+|\vec{b}|}\vec{a} + \frac{\vec{a}}{|\vec{a}|+|\vec{b}|}\vec{b} = \frac{|\vec{b}|\vec{a}+|\vec{a}|\vec{b}}{|\vec{a}|+|\vec{b}|}$$

$$|\vec{OC}| = \frac{||\vec{b}|\vec{a}+|\vec{a}|\vec{b}|}{|\vec{a}|+|\vec{b}|}$$

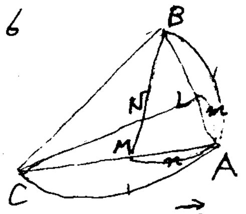
$$\therefore \quad \frac{|\vec{b}|\vec{a}+|\vec{a}|\vec{b}}{||\vec{b}|\vec{a}+|\vec{a}|\vec{b}|}$$

(2) $\vec{AB} = \vec{b}-\vec{a}$ AB方向の単位ベクトル $\dfrac{\vec{b}-\vec{a}}{|\vec{b}-\vec{a}|}$

$$\vec{OP} = \vec{a} + \frac{x(\vec{b}-\vec{a})}{|\vec{b}-\vec{a}|}$$

(3) $\vec{OC} = \dfrac{|\vec{b}|\vec{a}+|\vec{a}|\vec{b}}{|\vec{a}|+|\vec{b}|}$

3.6



$0<m<1$, $0<n<1$

$AB = \vec{a}$ $AC = \vec{b}$

(1) $\vec{BC} = \vec{b}-\vec{a}$ $\vec{BL} = (1-m)\cdot(-\vec{a}) = (m-1)\vec{a}$

$\vec{CM} = (n-1)\vec{b}$ $\vec{BM} = n\vec{b}-\vec{a}$ $\vec{CL} = m\vec{a}-\vec{b}$

(2) $BM : BN = 1 : s$ $CL : CN = 1 : t$

$BN = s\,BM$ $CN = t\,CL$

$\vec{BN} = sn\vec{b}-s\vec{a}$ $\vec{CN} = tm\vec{a}-t\vec{b}$

(3) $\vec{BC} = \vec{BN}+\vec{NC} = sn\vec{b}-s\vec{a}-tm\vec{a}+t\vec{b}$

$\therefore \vec{b}-\vec{a} = (sn+t)\vec{b}-(s+tm)\vec{a}$

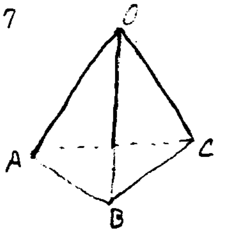
$\therefore 1 = sn+t.$ $1 = s+tm$

$n = sn+tmn$ $1-n = (1-mn)t$ $t = \dfrac{1-n}{1-mn}$

$$s = 1-tm = 1-\frac{m-mn}{1-mn} = \frac{1-m}{1-mn}$$

$$t = \frac{1-n}{1-mn} \quad s = \frac{1-m}{1-mn}$$

3.7



OA⊥BC , OB⊥AC ⇒ OC⊥AB

$\vec{OA}=\vec{a}$. $\vec{OB}=\vec{b}$. $\vec{OC}=\vec{c}$ とおくと

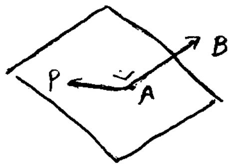
OA⊥BC より $\vec{a}\cdot(\vec{c}-\vec{b})=0$ ・・・①

OB⊥AC より $\vec{b}\cdot(\vec{c}-\vec{a})=0$ ・・②

∴ ①−② より $(\vec{a}-\vec{b})\cdot\vec{c}=0$ ∴ OC⊥AB

3.8 A(0.1.2) B(5.2.9) BA⊥S SはAを通る.

(1)



$\vec{AB}=(5,1,7)$

$(\vec{OB}-\vec{OA})\cdot(\vec{OP}-\vec{OA})=0$ $\vec{AP}\perp\vec{AB}$

(2) P(x.y.z) とおくと $\vec{AP}=(x,\ y-1,\ z-2)$

∴ $5x+y-1+7(z-2)=0$

$5x+y+7z-15=0$

3.9 (0. 0.1) (0.0,−1) からの距りの和が 2a $a>1$

(1) $\sqrt{x^2+y^2+(z-1)^2}+\sqrt{x^2+y^2+(z+1)^2}=2a$

$\dfrac{4z}{\sqrt{x^2+y^2+(z+1)^2}-\sqrt{x^2+y^2+(z-1)^2}}=2a$ $\sqrt{x^2+y^2+(z+1)^2}-\sqrt{x^2+y^2+(z-1)^2}=\dfrac{2z}{a}$

$2\sqrt{x^2+y^2+(z+1)^2}=2\left(a+\dfrac{z}{a}\right)$ $x^2+y^2+(z+1)^2=\left(a+\dfrac{z}{a}\right)^2$

$x^2+y^2+z^2=a^2+\dfrac{z^2}{a^2}-1$ $x^2+y^2+\dfrac{a^2-1}{a^2}z^2=a^2-1$

∴ $\dfrac{x^2}{a^2-1}+\dfrac{y^2}{a^2-1}+\dfrac{z^2}{a^2}=1$

(2) $\dfrac{4}{3}\pi a(a^2-1)$

3.10 $\vec{a}=(1,1,-1,1)$ $\vec{b}=(1,1,1,-1)$ $\vec{c}=(0,0,1,1)$

$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$ ∴ 互いに垂直

$\vec{\alpha}=(x,y,z,w)$ とおくと

$\begin{cases} x+y-z+w=0 \\ x+y+z-w=0 \\ z+w=0 \end{cases}$ $z=w$ ∴ $z=w=0$

∴ $\vec{\alpha}=(1,-1,0,0)$

§4 空間の基

4.1 $x=(x_1, x_2, \ldots, x_n)$ $y=(y_1, y_2, \ldots, y_n)$

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i = x \cdot {}^t y$$

$$A\vec{x} = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij} x_j\right) e_i$$

$$\langle A\vec{x}, \vec{y}\rangle = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij} x_j\right) y_i = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ij} y_i\right) x_j = \langle \vec{x}, {}^tA\vec{y}\rangle$$

$\therefore B = {}^tA$

4.2 (1) $R^4 \supset W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \middle| \begin{array}{l} x_1+x_2+x_3=0 \\ x_2+x_3+x_4=0 \end{array} \right\}$

${}^t(1, -1, 0, 1)$ ${}^t(1, 0, -1, 1)$

(2) 補空間のベクトルを (x, y, z, w) とすると

$$\begin{cases} x-y+w=0 \\ x-z+w=0 \end{cases}$$

$(1, 1, 1, 0)$, $(2, -1, -1, -3)$

$\frac{1}{\sqrt{3}}(1, 1, 1, 0)$, $\frac{1}{\sqrt{15}}(2, -1, -1, -3)$

4.3 $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$ $\begin{vmatrix} 1 & 4 & 3 \\ 2 & 3 & 4 \\ 3 & 2 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 3 \\ 0 & -5 & -2 \\ 0 & -10 & -3 \end{vmatrix} = \begin{vmatrix} 5 & 2 \\ 10 & 3 \end{vmatrix} \neq 0$

$\therefore$ 上の3ベクトルは一次独立 $\therefore$ R^3の基底である。

$$\begin{pmatrix} 8 \\ 11 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 3 & 4 \\ 3 & 2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 3 & 4 \\ 3 & 2 & 6 \end{pmatrix}^{-1}\begin{pmatrix} 8 \\ 11 \\ 4 \end{pmatrix}$$

x	y	z	
1	4	3	8
2	3	4	11
3	2	6	4
1	4	3	8
0	-5	-2	-5
0	-10	-3	-20
1	4	3	8
0	-5	-2	-5
0	0	1	-10
1	0	0	18
0	1	0	5
0	0	1	-10

$$\begin{cases} x = 18 \\ y = 5 \\ z = -10 \end{cases}$$

4.4 $W_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| \begin{array}{l} 2x_1 + 2x_2 + x_3 + 2x_4 = 0 \\ x_1 - x_2 - 2x_3 - 2x_4 = 0 \end{array} \right\}$

$W_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| \begin{array}{l} 9x_1 + 3x_2 + 2x_3 + x_4 = 0 \\ 3x_1 + x_2 - 6x_3 - x_4 = 0 \end{array} \right\}$

(1) $W_1 \cap W_2$

$$\mathrm{Rank}\begin{pmatrix} 2 & 2 & 1 & 2 \\ 1 & -1 & -2 & -2 \\ 9 & 3 & 2 & 1 \\ 3 & 1 & -6 & -1 \end{pmatrix} = \mathrm{Rank}\begin{pmatrix} 3 & 1 & -1 & 0 \\ 1 & -1 & -2 & -2 \\ 12 & 4 & -4 & 0 \\ 3 & 1 & -6 & -1 \end{pmatrix} = \mathrm{Rank}\begin{pmatrix} 3 & 1 & -1 & 0 \\ 1 & -1 & -2 & -2 \\ 0 & 0 & 0 & 0 \\ 3 & 1 & -6 & -1 \end{pmatrix}$$

$$= \mathrm{Rank}\begin{pmatrix} 3 & 1 & -1 & 0 \\ 4 & 0 & -3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & -1 \end{pmatrix} = \mathrm{Rank}\begin{pmatrix} 3 & 1 & -1 & 0 \\ 4 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & -1 \end{pmatrix} = 3$$

$W_1 \cup W_2$ の次元は 3 ∴ $W_1 \cap W_2$ の次元は 1次元

$3x_1 + x_2 - x_3 = 0$ $x_4 = -5x_3$

$4x_1 + 7x_3 = 0$ $x_1 = -\frac{7}{4}x_3$

$5x_3 + x_4 = 0$ $x_2 = x_3 + 3\frac{7}{4}x_3 = \frac{25}{4}x_3$

$x_3 = 4$ とおくと $\underline{(-7, 25, 4, -20)}$ 基底

(2) $W_1 + W_2$ の次元は3

4.5 (1) $A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$ $\mathrm{Rank}\,A = 3$

(2) $A x = 0$ $\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\left.\begin{array}{l} x_1 - x_2 = 0 \\ -x_2 + x_3 = 0 \\ x_4 - x_5 = 0 \end{array}\right\}$

$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

§5 行列

5.1 (1) $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$

(2) $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 7 \end{pmatrix}$

(3) $\begin{pmatrix} 6 & 4 \\ 4 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 4 & 8 & 9 \\ 2 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 32 & 72 & 74 \\ 20 & 44 & 46 \\ 14 & 30 & 32 \end{pmatrix}$

5.2 $A = \begin{pmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{pmatrix}$ $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$E-A = \begin{pmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{pmatrix}$ $E+A = \begin{pmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{pmatrix}$

$(E-A)^{-1} = \dfrac{1}{1+\tan^2\frac{\alpha}{2}}\begin{pmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{pmatrix} = \cos^2\frac{\alpha}{2}\begin{pmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{pmatrix}$

$(E-A)^{-1}(E+A) = \cos^2\frac{\alpha}{2}\begin{pmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{pmatrix}$

$= \cos^2\frac{\alpha}{2}\begin{pmatrix} 1-\tan^2\frac{\alpha}{2} & -2\tan\frac{\alpha}{2} \\ 2\tan\frac{\alpha}{2} & 1-\tan^2\frac{\alpha}{2} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

5.3 ${}^tA = A$ ${}^tB = B$

${}^t(AB) = {}^tB\,{}^tA = B\cdot A$ 、 $AB = B\cdot A$ のとき AB は対称

$AB \neq BA$ のとき AB は対称でない

5.4 (1) $f(x, y, z) = x^2+y^2+z^2-2xy-2yz$

$= x(x-y)+y(-x+y-z)+z(-y+z)$

$= (x\ y\ z)\begin{pmatrix} x-y \\ -x+y-z \\ -y+z \end{pmatrix} = (x, y\ z)\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\therefore A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

(2) $f(x,y,z) = (x, y, z)\, A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$ $\quad {}^tP\, P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\therefore \begin{pmatrix} u \\ v \\ w \end{pmatrix} = P \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ のとき $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = {}^tP \begin{pmatrix} u \\ v \\ w \end{pmatrix}$

$\therefore f(x, y, z) = (u, v, w)\, P A\, {}^tP \begin{pmatrix} u \\ v \\ w \end{pmatrix}$

$P A\, {}^tP = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1+\sqrt{2}}{2} & \frac{1-\sqrt{2}}{2} \\ 0 & -\frac{2+\sqrt{2}}{2} & \frac{\sqrt{2}-2}{2} \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{2}+1}{2} & \frac{1-\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}+1 & 0 \\ 0 & 0 & 1-\sqrt{2} \end{pmatrix}$

$\therefore (u, v, w)\, P A\, {}^tP \begin{pmatrix} u \\ v \\ w \end{pmatrix} = u^2 + (\sqrt{2}+1)v^2 + (1-\sqrt{2})w^2$

問5 $A = (a_{ij})$ $i > j$ のとき $a_{ij} = 0$ $B = (b_{ij})$ $i > j$ のとき $b_{ij} = 0$

$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & & a_{2n} \\ 0 & 0 & 0 & & a_{nn} \end{pmatrix}$

$A \cdot B = C = (c_{ij})$ とおくと

$i > j$ のとき $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{j} a_{ik} b_{kj}$ $\quad j \geqq k$

$\therefore a_{ij} = a_{ik} = 0$

$\therefore c_{ij} = 0.$

$\therefore A \cdot B = C$ は 上三角行列

5.6 $A = \begin{pmatrix} -2 & -1 & -2 & 1 \\ 5 & 3 & 4 & -1 \\ 1 & 0 & 1 & -1 \\ -3 & -1 & -2 & 2 \end{pmatrix}$ $A^2 - 3A = A(A - 3E)$

$$A - 3E = \begin{pmatrix} -5 & -1 & -2 & 1 \\ 5 & 0 & 4 & -1 \\ 1 & 0 & -2 & -1 \\ -3 & -1 & -2 & -1 \end{pmatrix}$$

$$A^2 - 3A = \begin{pmatrix} -2 & -1 & -2 & 1 \\ 5 & 3 & 4 & -1 \\ 1 & 0 & 1 & -1 \\ -3 & -1 & -2 & 2 \end{pmatrix} \begin{pmatrix} -5 & -1 & -2 & 1 \\ 5 & 0 & 4 & -1 \\ 1 & 0 & -2 & -1 \\ -3 & -1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 0 \\ -3 & -4 & -4 & -1 \\ -1 & -2 & -2 & 1 \\ 2 & 1 & 2 & -2 \end{pmatrix}$$

5.7 $C = \begin{pmatrix} 10 & 8 & 1 \\ 8 & 10 & 4 \\ 1 & 4 & 10 \end{pmatrix}$

$$C \circ C = \begin{pmatrix} \max(10.8.1) & \max(8.8.1) & \max(1.4.1) \\ \max(8.8.1) & \max(8.10.4) & \max(1.4.4) \\ \max(1.4.1) & \max(1.4.4) & \max(1.4.10) \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 8 & 4 \\ 8 & 10 & 4 \\ 4 & 4 & 10 \end{pmatrix}$$

$$C \circ C \circ C = \begin{pmatrix} 10 & 8 & 4 \\ 8 & 10 & 4 \\ 4 & 4 & 10 \end{pmatrix}$$

5.8 $f(x) = a_0 + a_1 x + a_2 x^2$

$V = \{ f(x) = a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in R \}$

$$f(x) \to \int_{-1}^{1} (t - x)^2 f(t)\,dt = \int_{-1}^{1} (t^2 - 2xt + x^2)(a_0 + a_1 t + a_2 t^2)\,dt$$

$$= \int_{-1}^{1} [(a_0 + a_1 t + a_2 t^2) x^2 - 2(a_0 t + a_1 t^2 + a_2 t^3) x + (a_0 t^2 + a_1 t^3 + a_2 t$$

$$= (2a_0 + \tfrac{2}{3} a_2) x^2 - 2(\tfrac{2}{3} a_1) x + \tfrac{2}{3} a_0 + \tfrac{2}{5} a_2$$

$$= \tfrac{2}{3} a_0 + \tfrac{2}{5} a_2 - \tfrac{4}{3} a_1 x + (2a_0 + \tfrac{2}{3} a_2) x^2$$

$$\begin{pmatrix} \frac{2}{3} & 0 & \frac{2}{5} \\ 0 & -\frac{4}{3} & 0 \\ 2 & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

$\therefore \ (a_0 \ a_1 \ a_2) \rightarrow (a_0 . a_1 \ a_2) \begin{pmatrix} \frac{2}{3} & 0 & 2 \\ 0 & -\frac{4}{3} & 0 \\ \frac{2}{5} & 0 & \frac{2}{3} \end{pmatrix}$

5.9 $a_0 + a_1 \sin x + b_1 \cos x + a_2 \sin 2x + b_2 \cos 2x$

(1) $V = \{ f(x) \mid f(x) = a_0 + a_1 \sin x + b_1 \cos x + a_2 \sin 2x + b_2 \cos 2x \}$

Vの基底は $1. \sin x \ \cos x , \sin 2x \ \cos 2x$, 5次元

(2) $\langle f . g \rangle = \int_0^{2\pi} f(t) g(t) \, dt$

$T : \ f(x) \longrightarrow f(x+\alpha)$

$$(a_0 \ a_1 \ b_1 \ a_2 \ b_2) \begin{pmatrix} 1 \\ \sin x \\ \cos x \\ \sin 2x \\ \cos 2x \end{pmatrix} \rightarrow (a_0 \ a_1 \ b_1 \ a_2 \ b_2) \begin{pmatrix} 1 \\ \sin(x+\alpha) \\ \cos(x+\alpha) \\ \sin(2x+2\alpha) \\ \cos(2x+2\alpha) \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ \sin(x+\alpha) \\ \cos(x+\alpha) \\ \sin(2x+2\alpha) \\ \cos(2x+2\alpha) \end{pmatrix} = \begin{pmatrix} 1 \\ \cos\alpha \sin x + \sin\alpha \cos x \\ -\sin\alpha \sin x + \cos\alpha \cos x \\ \cos 2\alpha \sin 2x + \sin 2\alpha \cos 2x \\ -\sin 2\alpha \sin 2x + \cos 2\alpha \cos 2x \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 0 & \cos 2\alpha & \sin 2\alpha \\ 0 & 0 & 0 & -\sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} 1 \\ \sin x \\ \cos x \\ \sin 2x \\ \cos 2x \end{pmatrix}$$

$$\therefore T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 0 & \cos 2\alpha & \sin 2\alpha \\ 0 & 0 & 0 & -\sin 2\alpha & \cos 2\alpha \end{pmatrix}$$

${}^tT . T = E \quad \therefore$ 直交変換

5.10

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 - x_2 \\ 4x_1 + 2x_2 - x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 4 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\therefore A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 4 & 2 & -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 4 & 2 & -1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 2 & -1 & -4 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 8 & -2 & -1 \end{array}\right)$$

$$\therefore B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 8 & -2 & -1 \end{pmatrix}$$

5.11

$$A = \begin{pmatrix} -1 & 0 & 0 \\ a & 1 & 0 \\ -ab & 0 & 1 \end{pmatrix} \qquad AX = \begin{pmatrix} -1 & 2 & 3 \\ -a & 0 & a \\ 0 & ab & ab \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 0 \\ a & 1 & 0 & 0 & 1 & 0 \\ -ab & 0 & 1 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & a & 1 & 0 \\ 0 & 0 & 1 & -ab & 0 & 1 \end{array}\right) \qquad \therefore A^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ a & 1 & 0 \\ -ab & 0 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} -1 & 0 & 0 \\ a & 1 & 0 \\ -ab & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & 3 \\ -a & 0 & a \\ 0 & ab & ab \end{pmatrix} = \begin{pmatrix} 1 & -2 & -3 \\ -2a & 2a & a^2 \\ ab & -ab & -2ab \end{pmatrix}$$

6.1 $A = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$ $\quad |A - \lambda E| = 0$

$(a-\lambda)^3 + 2b^3 - 3b^2(a-\lambda) = 0$

$(a-\lambda-b)^2(a-\lambda+2b) = 0$

$\begin{array}{cccc|c} 1 & 0 & -3b^2 & 2b^3 & b \\ & b & b^2 & -2b^3 & \\ \hline 1 & b & -2b^2 & 0 & \\ & b & 2b^2 & & \\ \hline 1 & 2b & 0 & & \end{array}$

$\therefore \lambda = a-b,\ a+2b$

$\lambda = a-b$ のとき

$\begin{pmatrix} b & b & b \\ b & b & b \\ b & b & b \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad p+q+r=0 \quad \therefore \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda = a+2b$

$\begin{pmatrix} -2b & b & b \\ b & -2b & b \\ b & b & -2b \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad \begin{array}{l} -2p+q+r=0 \\ p-2q+r=0 \\ p+q-2r=0 \end{array} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\therefore P = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ とおくと

$\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}$

$P^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}$

$P^{-1}AP = \begin{pmatrix} a-b & 0 & 0 \\ 0 & a-b & 0 \\ 0 & 0 & a+2b \end{pmatrix}$

$A^n = P \begin{pmatrix} (a-b)^n & 0 & 0 \\ 0 & (a-b)^n & 0 \\ 0 & 0 & (a+2b)^n \end{pmatrix} P^{-1}$

i) $|a-b|<1,\ |a+2b|<1$ のとき $\lim_{n\to\infty} A^n = 0$

ii) $a-b=1 \quad |a+2b|<1$ のとき

$\lim_{n\to\infty} A^n = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}P^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 0 & -3 & 3 \end{pmatrix}$

iii) $a-b=1 \quad a+2b=1$ のとき $(a=1,\ b=0)$

$\lim_{n\to\infty} A^n = E$

iv) $|a-b|<1 \quad |a+2b|=1$ のとき $\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ 他は発散

6.2 (1) $(\lambda I+A)^n=\lambda^n I+\sum_{k=1}^{n} {}_nC_k \lambda^{n-k}A^k$ であること

[I] $n=1$ のとき $(\lambda I+A)^1=\lambda I+A$ で成り立つ

[II] $n=m$ のとき成り立つと仮定すると

$$(\lambda I+A)^m=\lambda^m I+\sum_{k=1}^{m} {}_mC_{m-k}\lambda^{m-k}A^k$$

$n=m+1$ のとき

$$(\lambda I+A)^{m+1}=\left(\lambda^m I+\sum_{k=1}^{m} {}_mC_{m-k}\lambda^{m-k}A^k\right)(\lambda I+A)$$

$$=\lambda^{m+1}I+\sum_{k=1}^{m}\left({}_mC_{m-k}+{}_mC_{m-k+1}\right)\lambda^{m+1-k}A^k$$

$$=\lambda^{m+1}I+\sum_{k=1}^{m+1} {}_{m+1}C_k\lambda^{m+1-k}A^k$$

∴ $n=m+1$ のときも成り立つ

[I], [II] よりすべての n に対して成り立つ

(2) $A=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$ $A^2=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$ $A^3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$.

(3) $B=\begin{pmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{pmatrix}=\lambda I+A$

$B^n=\lambda^n I+\sum_{k=1}^{n} {}_nC_k\lambda^{n-k}A^k$ (2) より $A^n=0$, $n\geqq 3$

$$=\lambda^n I+{}_nC_1\lambda^{n-1}A+{}_nC_2\lambda^{n-2}A^2$$

$$=\begin{pmatrix}\lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2}\\ 0 & \lambda^n & n\lambda^{n-1}\\ 0 & 0 & \lambda^n\end{pmatrix}$$

6.3 $A=\begin{pmatrix}0&t\\-t&0\end{pmatrix}=t\begin{pmatrix}\cos\frac{\pi}{2}&\sin\frac{\pi}{2}\\-\sin\frac{\pi}{2}&\cos\frac{\pi}{2}\end{pmatrix}$ $A^n=t^n\begin{pmatrix}\cos\frac{n\pi}{2}&\sin\frac{n\pi}{2}\\-\sin\frac{n\pi}{2}&\cos\frac{n\pi}{2}\end{pmatrix}$

$n=2m$ のとき

$\cos\frac{n\pi}{2}=\cos m\pi=(-1)^m$ $\sin\frac{n\pi}{2}=\sin m\pi=0$

$n=2m-1$ のとき

$\cos\frac{n\pi}{2}=\cos(m-\frac{1}{2})\pi=0$ $\sin\frac{n\pi}{2}=\sin(m\pi-\frac{\pi}{2})=(-1)^{m-1}$

$$\sum_{n=1}^{\infty} x^n \cos\frac{n\pi}{2} = \sum_{m=1}^{\infty} x^{2m}\cos(m\pi) = \sum_{m=1}^{\infty}(-x^2)^m = \frac{-x^2}{1+x^2} \qquad |x|<1$$

$$\sum_{n=1}^{\infty} x^n \sin\frac{n\pi}{2} = \sum_{m=1}^{\infty} x^{2m-1}\sin(m\pi-\frac{\pi}{2}) = \sum_{m=1}^{\infty} x(-x^2)^{m-1} = \frac{x}{1+x^2} \qquad |x|<1$$

$$\therefore\ B = \begin{pmatrix} \frac{1}{1+x^2} & \frac{x}{1+x^2} \\ \frac{-x}{1+x^2} & \frac{x}{1+x^2} \end{pmatrix} \cdot |x|<1 \quad |x|\geq 1 \text{ のとき発散}$$

6.4 (1) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}$ $A^2 = \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$A^3 = \begin{pmatrix} \cos\frac{3\pi}{2} & -\sin\frac{3}{2}\pi \\ \sin\frac{3}{2}\pi & \cos\frac{3}{2}\pi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $A^4 = \begin{pmatrix} \cos 2\pi & -\sin 2\pi \\ \sin 2\pi & \cos 2\pi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(2) $$e^{\theta A} = \sum_{n=0}^{\infty} \frac{A^n}{n!}\theta^n$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\theta + \frac{\theta^2}{2!}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\theta^3}{3!}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{\theta^4}{4!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} \sum_{m=0}^{\infty}\frac{(-\theta^2)^m}{(2m)!} & \sum_{m=1}^{\infty}\frac{(-1)^m\theta^{2m-1}}{(2m-1)!} \\ \sum_{m=1}^{\infty}\frac{(-1)^{m-1}\theta^{2m-1}}{(2m-1)!} & \sum_{m=0}^{\infty}\frac{(-\theta^2)^m}{(2m)!} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

6.5 $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

(1) $A^2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ $A^3 = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ $A^4 = \begin{pmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$

$$A^n = \begin{pmatrix} 1 & {}_nC_1 & {}_{n+1}C_2 \\ 0 & 1 & {}_nC_1 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) $n=1$ のとき $A^1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & {}_1C_1 & {}_2C_2 \\ 0 & 1 & {}_1C_1 \\ 0 & 0 & 1 \end{pmatrix}$ ∴ 成り立つ

[I] $n=k$ のとき $A^k = \begin{pmatrix} 1 & {}_kC_1 & {}_{k+1}C_2 \\ 0 & 1 & {}_kC_1 \\ 0 & 0 & 1 \end{pmatrix}$ とすると

[II] $n=k+1$ のとき $A^{k+1} = \begin{pmatrix} 1 & 1+{}_kC_1 & 1+{}_kC_1+{}_{k+1}C_2 \\ 0 & 1 & 1+{}_kC_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & {}_{k+1}C_1 & {}_{k+2}C_2 \\ 0 & 1 & {}_{k+1}C_1 \\ 0 & 0 & 1 \end{pmatrix}$

∴ $n=k+1$ のとき成り立

[I],[II] より自然数 n に対して成り立つ

6.6 $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ $A^2 = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}$ $A^3 = \begin{pmatrix} 1 & 3a \\ 0 & 1 \end{pmatrix}$

$$A^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}$$

6.7 $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}$

$$A^{50} = \begin{pmatrix} \cos 25\pi & \sin 25\pi \\ -\sin 25\pi & \cos 25\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

6.8 $A = \begin{pmatrix} a & 1-a \\ 0 & 1 \end{pmatrix}$ $(0 < a < 1)$

a	$1-a$	1	0
0	1	0	1
1	0	$\frac{1}{a}$	$\frac{a-1}{a}$
0	1	0	1

(a) $A^{-1} = \frac{1}{a}\begin{pmatrix} 1 & a-1 \\ 0 & a \end{pmatrix}$

(b) $A^2 - (a+1)A + aI = \begin{pmatrix} a^2 & 1-a^2 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a^2+a & 1-a^2 \\ 0 & 1+a \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(c) (b) より

$A^2 = (a+1)A - aI$ $A^3 = (a+1)A^2 - aA = (a+1)\{(a+1)A - aI\} - aA$

$$= (a^2+a+1)A - (a^2+a)I$$

$A^4 = (a^2+a+1)A^2 - (a^2+a)A = (a^2+a+1)\{(a+1)A - aI\} - (a^2+a)A$

$$= (a^3+a^2+a+1)A - (a^3+a^2+a)I$$

$A^n = (a^{n-1}+a^{n-2}+\cdots+a+1)A - (a^{n-1}+a^{n-2}+\cdots+a)I$

$$= \frac{1-a^n}{1-a}A - \frac{a(1-a^{n-1})}{1-a}I$$

$\lim_{n\to\infty} A^n = \frac{1}{1-a}A - \frac{a}{1-a}I = \begin{pmatrix} \frac{a}{1-a} & 1 \\ 0 & \frac{1}{1-a} \end{pmatrix} - \begin{pmatrix} \frac{a}{1-a} & 0 \\ 0 & \frac{a}{1-a} \end{pmatrix}$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

6.9 $A = \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}$

(1) $A\begin{pmatrix} x \\ -x \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}\begin{pmatrix} x \\ -x \end{pmatrix} = \begin{pmatrix} x \\ (\alpha-\beta)x \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix}$

$\alpha - \beta = -1$ $\therefore \beta - \alpha = 1$

(2) $A^2 - 3A + 2E = 0$. $A^2 = \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha+\alpha\beta & \beta^2 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ \alpha+\alpha\beta & \beta^2 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 3\alpha & 3\beta \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -2\alpha+\alpha\beta & \beta^2-3\beta+2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\beta^2 - 3\beta + 2 = 0$ $(\beta-1)(\beta-2) = 0$ $\beta = 1.\ 2$ $\alpha(\beta-2) = 0$ $\alpha = 0.\ \beta = 2$

$\therefore \beta = 2$, $\beta = 1$

(3) (2) より $\beta = 2$ (1) に代入して $\alpha = 1$

$\therefore A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$

$\beta = 1,\ \alpha = 0$

$A = E$

$A^n = E$

$A^2 = 3A - 2E = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$

$A^3 = A^2 A = 3A^2 - 2A = 3(3A-2E) - 2A$

$= 7A - 6E = \begin{pmatrix} 7 & 0 \\ 7 & 14 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 7 & 8 \end{pmatrix}$

$A^4 = A^3 A = 7A^2 - 6A = 7(3A-2E) - 6A$

$= 15A - 14A = \begin{pmatrix} 15 & 0 \\ 15 & 30 \end{pmatrix} - \begin{pmatrix} 14 & 0 \\ 0 & 14 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 15 & 16 \end{pmatrix}$

$A^n = \begin{pmatrix} 1 & 0 \\ 2^n - 1 & 2^n \end{pmatrix}$

6.10 $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

(1) $A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $A^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $A^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $n \geqq 3$

(2) $B^2 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ $B^3 = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ $A^n = \begin{pmatrix} 1 & {}_nC_1 & {}_nC_2 \\ 0 & 1 & {}_nC_1 \\ 0 & 0 & 1 \end{pmatrix}$

$AD-BC=1,\quad A+D=2\cos\theta$

$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}$ とおくとき

$A_n \sin\theta = A\sin n\theta - \sin(n-1)\theta \qquad B_n \sin\theta = B\sin n\theta$

$C_n \sin\theta = C\sin n\theta \qquad D_n \sin\theta = D\sin n\theta - \sin(n-1)\theta$

であることを数学的帰納法で証明する。

[I] $n=1$ のとき $A_1=A,\ B_1=B\ \ C_1=C\ \ D_1=D$ で成り立つ

$n=2$ のとき

$\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^2 = \begin{pmatrix} A^2+BC & AB+BD \\ AC+CD & CB+D^2 \end{pmatrix}$ 条件より

$= \begin{pmatrix} A(A+D)-1 & B(A+D) \\ C(A+D) & D(A+D)-1 \end{pmatrix} = \begin{pmatrix} 2A\cos\theta-1 & 2B\cos\theta \\ 2C\cos\theta & 2D\cos\theta-1 \end{pmatrix}$

$\therefore A_2\sin\theta = A\sin 2\theta - \sin\theta \qquad B_2\sin\theta = B\sin 2\theta$

$C_2\sin\theta = C\sin 2\theta \qquad D_2\sin\theta = D\sin 2\theta - \sin\theta$

[II] $n=m$ のとき成り立つと仮定すると

$n=m+1$ のとき

$\begin{pmatrix} A_{m+1} & B_{m+1} \\ C_{m+1} & D_{m+1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A_m & B_m \\ C_m & D_m \end{pmatrix} = \begin{pmatrix} AA_m+BC_m & AB_m+BD_m \\ CA_m+DC_m & CB_m+DD_m \end{pmatrix}$

$A_{m+1}\sin\theta = AA_m\sin\theta + BC_m\sin\theta = A^2\sin m\theta - A\sin(m-1)\theta + BC\sin m\theta$

$=\{A(A+D)-1\}\sin m\theta - A\sin(m-1)\theta$

$= 2A\cos\theta\sin m\theta - \sin m\theta - A\sin(m-1)\theta$

$= A\{\sin(m+1)\theta + \sin(m-1)\theta\} - A\sin(m-1)\theta - \sin m\theta$

$= A\sin(m+1)\theta - \sin m\theta$

$B_{m+1}\sin\theta = AB_m\sin\theta + BD_m\sin\theta = AB\sin m\theta + B(D\sin m\theta - \sin(m-1)\theta)$

$= B\cdot 2\cos\theta\sin m\theta - B\sin(m-1)\theta = B\{\sin(m+1)\theta + \sin(m-1)\theta\} - B\sin(m-1)\theta$

$= B\sin(m+1)\theta$

同様に $C_{m+1}\sin\theta = C\sin(m+1)\theta \qquad D_m\sin\theta = D\sin(m+1)\theta - \sin m\theta$

$\therefore n=m+1$ のとき成り立つ

[I].[II] よりすべての正整数に対して成り立つ

§7 行列と図形

7.1 $ax^2+2bxy+cy^2=1 \qquad (x\ y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=1$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$

$$(X\ Y)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}=1$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a\cos\theta-b\sin\theta & a\sin\theta+b\cos\theta \\ b\cos\theta-c\sin\theta & b\sin\theta+c\cos\theta \end{pmatrix}$$

$$=\begin{pmatrix} a\cos^2\theta-2b\sin\theta\cos\theta+c\sin^2\theta & a\sin\theta\cos\theta-c\sin\theta\cos\theta+2b(\cos^2\theta-\sin^2\theta) \\ a\sin\theta\cos\theta-c\sin\theta\cos\theta+b(\cos^2\theta-\sin^2\theta) & a\sin^2\theta+c\cos^2\theta+2b\sin\theta\cos\theta \end{pmatrix}$$

$$\therefore\ \frac{a-c}{2}\sin 2\theta+b\cos 2\theta=0 \quad \therefore\ \tan 2\theta=\frac{-2b}{a-c}$$

7.2 $A\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \therefore A=\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

7.3 (1) $A=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$${}^tA A=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

∴ Aは直交行列

(2) 原点を中心として θだけ回転する一次変換

7.4 7.1において

$A=\frac{a-b}{2}\cos 2\theta-h\sin 2\theta+\frac{a+b}{2} \qquad H=\frac{a-b}{2}\sin 2\theta+h\cos 2\theta$

$B=-\frac{a-b}{2}\cos 2\theta+h\sin 2\theta+\frac{a+b}{2}$

$A+B=a+b \qquad AB-H^2=\frac{(a+b)^2}{4}-\frac{(a-b)^2}{4}-h^2=ab-h^2$

7.5 (a) ${}^t\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\neq\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ∴ 直交行列でない

(b) ${}^t\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ∴ 直交行列

$\cos\frac{\pi}{3}=\frac{1}{2} \quad \sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$ $\qquad \frac{\pi}{3}$ 回転

(c) ${}^t\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ∴ 直交でない

(d) ${}^t\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ∴ 直交

x軸に関して対称に移動

7.6 $P = \begin{pmatrix} \frac{1}{2} & a \\ b & \frac{1}{2} \end{pmatrix}$ A(1, 0) B(0, 1)

(1) $\vec{OA'}$: $\begin{pmatrix} \frac{1}{2} & a \\ b & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ b \end{pmatrix}$ $\vec{OB'}$: $\begin{pmatrix} \frac{1}{2} & a \\ b & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ \frac{1}{2} \end{pmatrix}$

$\cos\theta' = \dfrac{\frac{1}{2}(a+b)}{\frac{1}{4}\sqrt{1+4a^2}\sqrt{1+4b^2}} = \dfrac{2(a+b)}{\sqrt{1+4a^2}\sqrt{1+4b^2}}$

(2) $\triangle OA'B' = \frac{1}{2}\sqrt{\frac{1}{16}(1+4a^2)(1+4b^2) - \frac{1}{4}(a+b)^2}$

$= \frac{1}{8}\sqrt{(4ab-1)^2} = \frac{1}{8}|4ab-1|$

$S = 0$ となるには $4ab-1=0$ ∴ $|P| = 0$

(3) $\triangle OAB$ と $\triangle O'A'B'$ が合同のとき $\theta = \frac{\pi}{2}$ ∴ $\cos\theta' = 0$

∴ $a+b=0$ $b=-a$

$\triangle OAB = \triangle OA'B'$ ∴ $\frac{1}{8}|4ab-1| = \frac{1}{2}$ ∴ $|-4a^2-1| = 4$

∴ $4a^2 = 3$ $a = \frac{\sqrt{3}}{2}$ $b = -\frac{\sqrt{3}}{2}$ ($a>0$ より)

7.7 A(0, 1, 2) B(5, 2, 9)

$\vec{AB} = (5, 1, 7)$

(1) S : $5x + (y-1) + 7(z-2) = 0$ Q$(x, 0, 0)$

$5x = 15$ $x = 3$ ∴ Q(3, 0, 0)

$\vec{OA}\cdot\vec{OQ} = 0+0+0=0$ ∴ $OA \perp OQ$

(2) 求める点を T(x, y, z) とすると

$OT \perp OA$ $OT = OA$, Tは S 上にある

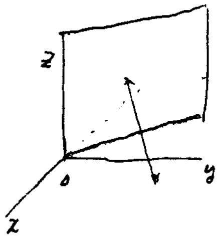
$y+2z=0$ $5x+y+7z=15$ $x^2+y^2+z^2=9$

$y=-2z$ $x=3-z$ $(3-z)^2+4z^2+z^2=9$

$6z^2-6z=0$ $z=0, 1$ $x=3, 2$ $y=0, -2$

∴ (3, 0, 0) or (2, -2, 1)　∴ T(2, -2, 1)

7.8 (1)



点 (x, y, z) と 平面 x+y=0 に関して対称な点を (x', y', z') とすると

$z' = z$　　$\left(\frac{x+x'}{2}, \frac{y+y'}{2}, \frac{z+z'}{2}\right)$ は平面上にある

∴ $\frac{x+x'}{2} + \frac{y+y'}{2} = 0$

$(x-x', y-y', z-z')$ は平面に垂直

∴ $\frac{y-y'}{x-x'} = 1$

∴ $z = z'$

$x'+y' = -x-y$

$x'-y' = x-y$

$x' = -y,\quad y' = -x$

∴ $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$　∴ $A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(2) P(x, y, z) を平面 x+y+z=0 に関して対称に移した点を P'(x', y', z') とする

$\frac{x+x'}{2} + \frac{y+y'}{2} + \frac{z+z'}{2} = 0$

$PP' \perp \pi$　∴ $(x'-x, y'-y, z'-z) = t(1, 1, 1)$

∴ $x+y+z+x'+y'+z' = 0$　　$x' = x+t \quad y' = y+t, \quad z' = z+t$

∴ $2(x+y+z) + 3t = 0$.　∴ $t = -\frac{2}{3}(x+y+z)$

∴ $x' = \frac{1}{3}(x-2y-2z)$

$y' = \frac{1}{3}(-2x+y-2z)$

$z' = \frac{1}{3}(-2x-2y+z)$

∴ $B = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$

§8 逆行列

8.1 (1) $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -2 \neq 0$ ∴ 正則

逆行列 $\frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

(2) $\begin{vmatrix} 2 & 2 & -3 \\ 1 & -1 & 2 \\ 3 & 0 & 4 \end{vmatrix} = -13 \neq 0$ ∴ 正則

逆行列 $\frac{1}{13}\begin{pmatrix} 4 & 8 & -1 \\ -2 & -17 & 7 \\ -3 & -6 & 4 \end{pmatrix}$

(3) $\begin{vmatrix} 1 & 1 & 2 \\ 0 & a & -1 \\ 0 & 1 & 1 \end{vmatrix} = a+1$ $a \neq -1$ のとき 正則

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & a & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & a+1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & \frac{1}{a+1} & -\frac{2a+1}{a+1} \\ 0 & a+1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{a+1} & \frac{a}{a+1} \end{array}\right)$$

∴ 逆行列 $\frac{1}{a+1}\begin{pmatrix} a+1 & 1 & -2a-1 \\ 0 & 1 & 1 \\ 0 & -1 & a \end{pmatrix}$

(4) $\begin{vmatrix} 6 & 2 & 7 \\ 4 & 1 & 5 \\ 7 & 1 & 8 \end{vmatrix} = 3 \neq 0$ 正則

逆行列 $\frac{1}{3}\begin{pmatrix} 3 & -9 & 3 \\ 3 & -1 & -2 \\ -3 & 8 & -2 \end{pmatrix}$

(5) $\begin{vmatrix} -5 & 3 \\ 3 & -2 \end{vmatrix} = 1 \neq 0$ 正則

逆行列 $\begin{pmatrix} -2 & -3 \\ -3 & -5 \end{pmatrix}$

8.2 (1) $\begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 2 & -2 \end{vmatrix} = 8$ ∴ 正則で 逆行列 $\frac{1}{8}\begin{pmatrix} 0 & 4 & 4 \\ 4 & -5 & -3 \\ 4 & -3 & -5 \end{pmatrix}$

(2) $\begin{vmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{vmatrix} = 27$ 逆行列 $\frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$

(3)

$$\left|\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & -1 & 0 & 1 \\ \hline 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right|$$

逆行列 $\frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

(4) $\begin{vmatrix} 1 & 5 & 7 \\ 8 & 3 & 4 \\ 2 & 5 & 0 \end{vmatrix} = 258$ 逆行列 $\frac{1}{258}\,{}^t\!\begin{pmatrix} \begin{vmatrix} 3 & 4 \\ 5 & 0 \end{vmatrix} & -\begin{vmatrix} 8 & 4 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 8 & 3 \\ 2 & 5 \end{vmatrix} \\ -\begin{vmatrix} 5 & 7 \\ 5 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 7 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 5 \\ 2 & 5 \end{vmatrix} \\ \begin{vmatrix} 5 & 7 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 7 \\ 8 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 8 & 3 \end{vmatrix} \end{pmatrix}$

$$= \frac{1}{258}\begin{pmatrix} -20 & 35 & -1 \\ 8 & -14 & 52 \\ 34 & 5 & -37 \end{pmatrix}$$

(5) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix}^{-1} = \frac{1}{16}\begin{pmatrix} -4 & 4 & 4 \\ 4 & -8 & 4 \\ 4 & 4 & -4 \end{pmatrix}$

(6) $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 4 & 2 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

(7) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

(8) $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$

(9) $\begin{pmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 4 \end{pmatrix}$

8.) $\begin{vmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 & 1 \\ 3 & 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \alpha \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1-\alpha \\ 0 & 0 & 0 & 0 & \alpha \end{vmatrix} = \alpha \neq 0$

8.4 $|A| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ a & b-a & c-a & d-a \\ a^2 & b^2-a^2 & c^2-a^2 & d^2-a^2 \\ a^3 & b^3-a^3 & c^3-a^3 & d^3-a^3 \end{vmatrix} = (b-a)(c-a)(d-a) \begin{vmatrix} 1 & 1 & 1 \\ b+a & c+a & d+a \\ b^2+ab+a^2 & c^2+ca+a^2 & d^2+ad+a^2 \end{vmatrix}$

$= (b-a)(c-a)(d-a) \begin{vmatrix} 1 & 0 & 0 \\ b+a & c-b & d-b \\ b^2+ba+a^2 & c^2-b^2+a(c-b) & d^2-b^2+a(d-b) \end{vmatrix}$

$= (b-a)(c-a)(d-a)(c-b)(d-b) \begin{vmatrix} 1 & 1 \\ a+b+c & d+b+a \end{vmatrix}$

$= (b-a)(c-a)(d-a)(c-d)(d-b)(d-c)$

a, b, c, d のどの2つも等しくない

8.5 $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad A^2 = \begin{pmatrix} a^2+bc & ab+db \\ ac+cd & bc+d^2 \end{pmatrix}$

(1) $A^2 - (a+d)A + (ad-bc)E$

$= \begin{pmatrix} a^2+bc & ab+db \\ ac+cd & bc+d^2 \end{pmatrix} - \begin{pmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(2) (1) より $(ad-bc)E = A((a+d)E - A)$

$\therefore ad-bc \neq 0$

(3) (1), (2) より

$A^{-1} = \dfrac{1}{ad-bc} \left\{ \begin{pmatrix} a+d & 0 \\ 0 & a+d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$

$= \dfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

8.6 $A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\quad A(\theta)^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = A(-\theta)$

$\therefore A(\theta)^{-1} = A(-\theta)$

8.7 $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ $\quad A^2 = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}$ $\quad A^3 = \begin{pmatrix} 1 & 3a \\ 0 & 1 \end{pmatrix}$

(1) $A^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}$

(2) $S = \sum_{k=1}^{n} A^k = \begin{pmatrix} n & \frac{n(n+1)}{2}a \\ 0 & n \end{pmatrix}$

(3) $S^{-1} = \frac{1}{n^2} \begin{pmatrix} n & -\frac{n(n+1)}{2}a \\ 0 & n \end{pmatrix}$

8.8 A, B が正則のとき $|A| \neq 0$, $|B| \neq 0$ ∴ $|AB| = |A||B| \neq 0$

∴ AB は正則

$AB(B^{-1}A^{-1}) = E$ ∴ $(AB)^{-1} = B^{-1}A^{-1}$

8.9 $|A| = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}$

$= (b-a)(c-a)(c-b)$

$A^{-1} = \frac{1}{(b-a)(c-a)(c-b)} \begin{pmatrix} bc(c-b) & b^2-c^2 & c-b \\ ac(a-c) & c^2-a^2 & a-c \\ ab(b-a) & a^2-b^2 & b-a \end{pmatrix}$

8.11 $A = \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}$ $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$AB = I$ $B = A^{-1} = \frac{1}{7} \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$

§ 9 行列式

9.1 (1)

$$\begin{vmatrix} 1 & a & b & c+d \\ 1 & b & c & d+a \\ 1 & c & d & a+b \\ 1 & d & a & b+c \end{vmatrix} = \begin{vmatrix} 1 & a & b & c+d \\ 0 & b-a & c-b & a-c \\ 0 & c-b & d-c & b-d \\ 0 & d-c & a-d & c-a \end{vmatrix} = \begin{vmatrix} b-a & a-b & a-c \\ c-b & b-c & b-d \\ d-c & c-d & c-a \end{vmatrix} = 0$$

(2)

$$\begin{vmatrix} a+b+c & a+b & a & a \\ a+b & a+b+c & a & a \\ a & a & a+b+c & a+b \\ a & a & a+b & a+b+c \end{vmatrix} = \begin{vmatrix} a+b+c & -c & a & 0 \\ a+b & c & a & 0 \\ a & 0 & a+b+c & -c \\ a & 0 & a+b & c \end{vmatrix}$$

$$= \begin{vmatrix} c & -2c & 0 & 0 \\ a+b & c & a & 0 \\ 0 & 0 & c & -2c \\ a & 0 & a+b & c \end{vmatrix} = c\begin{vmatrix} c & a & 0 \\ 0 & c & -2c \\ 0 & a+b & c \end{vmatrix} + 2c\begin{vmatrix} a+b & a & 0 \\ 0 & c & -2c \\ a & a+b & c \end{vmatrix}$$

$$= c\{c^3+2c^2(a+b)\} + 2c\{c^2(a+b) - 2ca^2 + 2c(a+b)^2\}$$

$$= c^2\{c^2+2ca+2cb+2ca+2cb-4a^2+4a^2+8ab+4b^2\}$$

$$= c^2\{c^2+4c(a+b)+4b(b+2a)\}$$

$$= c^2(c+2b)(c+2b+4a)$$

(3)

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix} = (y-x)(z-x)\begin{vmatrix} 1 & 1 \\ y+x & z+x \end{vmatrix}$$

$$= (y-x)(z-x)(z-y)$$

(4)

$$\begin{vmatrix} 3 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 3\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 9$$

(5)

$$\begin{vmatrix} 2 & -1 & 1 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 & -1 \\ -1 & 1 & 1 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 \\ -1 & 0 & 3 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} = 7$$

(6) $\begin{vmatrix} a & 0 & 0 & b \\ 0 & 0 & c & 0 \\ 0 & d & 0 & 0 \\ e & 0 & 0 & -a \end{vmatrix} = a\begin{vmatrix} 0 & c & 0 \\ d & 0 & 0 \\ 0 & 0 & -a \end{vmatrix} - b\begin{vmatrix} 0 & 0 & c \\ 0 & d & 0 \\ e & 0 & 0 \end{vmatrix} = a^2cd + bcde$

$= cd(a^2 + be)$

(7) $\begin{vmatrix} -1 & 2 & 4 \\ 3 & -1 & 2 \\ 2 & 5 & 3 \end{vmatrix} = 71$

(8) $\begin{vmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & b_3 & b_2 \\ -a_2 & -b_3 & 0 & b_1 \\ -a_3 & -b_2 & -b_1 & 0 \end{vmatrix} = a_1\begin{vmatrix} a_1 & a_2 & a_3 \\ -b_3 & 0 & b_1 \\ -b_2 & -b_1 & 0 \end{vmatrix} - a_2\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_3 & b_2 \\ -b_2 & -b_1 & 0 \end{vmatrix} + a_3\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_3 & b_2 \\ -b_3 & 0 & b_1 \end{vmatrix}$

$= a_1(b_1 b_3 a_3 - b_2 a_2 b_1 + b_1^2 a_1) - a_2(-b_2^2 a_2 + b_2 b_3 a_3)$

$+ a_3(a_1 b_1 b_3 - a_2 b_2 b_3 + a_3 b_3^2)$

$= a_1^2 b_1^2 + a_2^2 b_2^2 + a_3^2 b_3^2 - 2(a_1 a_2 b_1 b_2 + a_1 a_3 b_1 b_3 - a_3 a_1 b_3 b_1)$

(9) $\begin{vmatrix} 3 & 1 & 1 & 1 \\ 2 & 1 & 2 & 3 \\ 7 & 2 & 3 & 5 \\ 2 & 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ -1 & 1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 3 & 2 & 6 \\ -1 & 1 & 1 & 3 \end{vmatrix}$

$= -\begin{vmatrix} 0 & 0 & -1 \\ 0 & 2 & 6 \\ -1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 2 & 6 \end{vmatrix} = 2$

(10) $\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{6} \end{vmatrix} = \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{12} & \frac{1}{12} \\ 0 & \frac{1}{12} & \frac{1}{18} \end{vmatrix} = \frac{1}{12\times18} - \frac{1}{12\times12} = \frac{2-3}{12\times6^2} = \frac{-1}{432}$

(11) $\begin{vmatrix} x & a & b & c \\ a & x & b & c \\ a & b & x & c \\ a & b & c & x \end{vmatrix} = \begin{vmatrix} x & a & b & c \\ a & x & b & c \\ 0 & b-x & x-b & 0 \\ 0 & b-x & c-b & x-c \end{vmatrix} = \begin{vmatrix} x-a & a-x & 0 & 0 \\ a & x & b & c \\ 0 & b-x & x-b & 0 \\ 0 & b-x & c-b & x-c \end{vmatrix}$

$= (x-a)\left\{\begin{vmatrix} x & b & c \\ b-x & x-b & 0 \\ b-x & c-b & x-c \end{vmatrix} + \begin{vmatrix} a & b & c \\ 0 & x-b & 0 \\ 0 & c-b & x-c \end{vmatrix}\right\}$

$$(x-a)(x-b)\left\{\begin{vmatrix} x & b & c \\ -1 & 1 & 0 \\ b-x & c-b & x-c \end{vmatrix}+\begin{vmatrix} a & b & c \\ 0 & 1 & 0 \\ 0 & c-b & x-c \end{vmatrix}\right.$$

$$=(x-a)(x-b)\left\{\begin{vmatrix} x+b & b & c \\ 0 & 1 & 0 \\ c-x & c-b & x-c \end{vmatrix}+\begin{vmatrix} a & c \\ 0 & x-c \end{vmatrix}\right.$$

$$=(x-a)(x-b)\{(x+b)(x-c)+c(x-c)+a(x-c)\}$$

$$=(x-a)(x-b)(x-c)(x+a+b+c)$$

(12) $$\begin{vmatrix} 3 & 2 & 7 & 5 \\ 4 & 0 & 4 & 8 \\ 8 & 0 & 2 & 3 \\ 7 & 5 & 1 & 6 \end{vmatrix}=\begin{vmatrix} -4 & 2 & 7 & -9 \\ 0 & 0 & 4 & 0 \\ 6 & 0 & 2 & -1 \\ 6 & 5 & 1 & 4 \end{vmatrix}=-4\begin{vmatrix} -4 & 2 & -9 \\ 6 & 0 & -1 \\ 6 & 5 & 4 \end{vmatrix}$$

$$=-4\begin{vmatrix} -4 & 2 & -9 \\ 6 & 0 & -1 \\ 0 & 5 & 5 \end{vmatrix}=-4\begin{vmatrix} -4 & 2 & -11 \\ 6 & 0 & -1 \\ 0 & 5 & 0 \end{vmatrix}$$

$$=20\begin{vmatrix} -4 & -11 \\ 6 & -1 \end{vmatrix}=1400$$

(13) $$\begin{vmatrix} a^2+1 & ab & ac & ad \\ ba & b^2+1 & bc & bd \\ ca & cb & c^2+1 & cd \\ da & db & dc & d^2+1 \end{vmatrix}=a^2\begin{vmatrix} 1+\frac{1}{a^2} & b & c & d \\ b & b^2+1 & bc & bd \\ c & cb & c^2+1 & cd \\ d & db & dc & d^2+1 \end{vmatrix}$$

$$=a^2\begin{vmatrix} 1+\frac{1}{a^2} & -\frac{b}{a^2} & -\frac{c}{a^2} & -\frac{d}{a^2} \\ b & 1 & 0 & 0 \\ c & 0 & 1 & 0 \\ d & 0 & 0 & 1 \end{vmatrix}$$

$$=a^2\begin{vmatrix} 1+\frac{1}{a^2}+\frac{b^2}{a^2}+\frac{c^2}{a^2}+\frac{d^2}{a^2} & -\frac{b}{a^2} & -\frac{c}{a^2} & -\frac{d}{a^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$=1+a^2+b^2+c^2+d^2$$

(14) $$\begin{vmatrix} x-1 & -2 & -3 & -4 \\ -1 & x-2 & -3 & -4 \\ -1 & -2 & x-3 & -4 \\ -1 & -2 & -3 & x-4 \end{vmatrix}=\begin{vmatrix} x & 0 & 0 & -x \\ 0 & x & 0 & -x \\ 0 & 0 & x & -x \\ -1 & -2 & -3 & x-4 \end{vmatrix}=\begin{vmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ -1 & -2 & -3 & x-10 \end{vmatrix}=x^3(x-10)$$

9.2 $x+y+z=\pi$ のとき

(1)

$$\begin{vmatrix} -1 & \cos z & \cos y \\ \cos z & -1 & \cos x \\ \cos y & \cos x & -1 \end{vmatrix} = -1 + 2\cos x\cos y\cos z + \cos^2 x + \cos^2 y + \cos^2 z$$

$$\cos x\cos y = \frac{1}{2}\{\cos(x+y) + \cos(x-y)\}$$

$$\cos z = \cos(\pi-(x+y)) = -\cos(x+y)$$

$$= -1 - \{\cos(x+y) + \cos(x-y)\}\cos(x+y) + \cos^2 x + \cos^2 y + \cos^2(x+y)$$

$$= -1 - \cos(x-y)\cos(x+y) + \cos^2 x + \cos^2 y$$

$$= -1 - \frac{1}{2}\{\cos 2x + \cos 2y\} + \cos^2 x + \cos^2 y$$

$$= -1 - \frac{1}{2}\{2\cos^2 x - 1 + 2\cos^2 y - 1\} + \cos^2 x + \cos^2 y = 0$$

(2) $\omega^3 = 1$ $\omega \neq 1$ のとき.

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & \omega^3 \\ \omega^2 & \omega^3 & \omega^4 \end{vmatrix} = \begin{vmatrix} 1 & \omega & \omega^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

9.3 (1)

$$\begin{vmatrix} -c & -a & a+b+c \\ -b & a+b+c & -a \\ a+b+c & -b & -c \end{vmatrix} = \begin{vmatrix} b & b+c & a+b+c \\ c & b+c & -a \\ a & -b-c & -c \end{vmatrix}$$

$$= \begin{vmatrix} a+b & 0 & a+b \\ a+c & 0 & -(a+c) \\ a & -(b+c) & -c \end{vmatrix} = (a+b)(a+c)(b+c)\begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ a & -1 & -c \end{vmatrix}$$

$$= (a+b)(b+c)(c+a)\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2(a+b)(b+c)(c+a)$$

(2)

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & b-a & b^2-a^2 & b^3-a^3 \\ 0 & c-a & c^2-a^2 & c^3-a^3 \\ 0 & d-a & d^2-a^2 & d^3-a^3 \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)\begin{vmatrix} 1 & b+a & b^2+ab+a^2 \\ 1 & c+a & c^2+ac+a^2 \\ 1 & d+a & d^2+ad+a^2 \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)\begin{vmatrix} 1 & b+a & b^2+ab+b^2 \\ 0 & c-b & c^2-b^2+a(c-b) \\ 0 & d-b & d^2-b^2+a(d-b) \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)\begin{vmatrix} 1 & c+b+a \\ 1 & d+b+a \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$$

(3) $\begin{vmatrix} 1+x^4 & x+x^3 & x^2 \\ 1+y^4 & y+y^3 & y^2 \\ 1+z^4 & z+z^3 & z^2 \end{vmatrix} = \begin{vmatrix} 1+x^4 & x+x^3 & x^2 \\ y^4-x^4 & y-x+y^3-x^3 & y^2-x^2 \\ z^4-x^4 & z-x+z^3-x^3 & z^2-x^2 \end{vmatrix}$

$= (y-x)(z-x) \begin{vmatrix} 1+x^4 & x+x^3 & x^2 \\ y^3+y^2x+yx^2+x^3 & 1+y^2+xy+x^2 & y+x \\ z^3+z^2x+zx^2+x^3 & 1+z^2+zx+x^2 & z+x \end{vmatrix} =$

$= (y-x)(z-x)(z-y) \begin{vmatrix} 1+x^4 & x+x^3 & x^2 \\ y^3+y^2x+yx^2+x^3 & 1+y^2+yx+x^2 & y+x \\ y^2+yz+z^2+xy+zx+x^2 & z+y+x & 1 \end{vmatrix} = (y-x)(z-x)(z-y) \begin{vmatrix} 1-x^2 & x & x^2 \\ y^3-x & 1+y^2 & y+x \\ y^2+yz+z^2 & z+y & 1 \end{vmatrix}$

$= (y-x)(z-x)(z-y) \begin{vmatrix} 1 & x & x^2 \\ y^3+y & 1+y^2 & y+x \\ y^2+yz+z^2+1 & z+y & 1 \end{vmatrix}$

$= (y-x)(z-x)(z-y)\{1-xy-yz-zx+xyz(x+y+z)-x^2y^2z^2\}$

(4) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & x & x^2 \\ 1 & x^3 & x^4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & x-1 & x^2-x \\ 1 & x^3-1 & x^4-x^3 \end{vmatrix} = (x-1)^2 \begin{vmatrix} 1 & x \\ x^2+x+1 & x^3 \end{vmatrix} = -x(x-1)^2(x+1)$

(5) $\begin{vmatrix} x-2 & -1 & 1 \\ 1 & x-4 & 1 \\ 1 & -1 & x-2 \end{vmatrix} = \begin{vmatrix} x-3 & -1 & 1 \\ 0 & x-4 & 1 \\ 3-x & -1 & x-2 \end{vmatrix} = \begin{vmatrix} x-3 & -1 & 1 \\ 0 & x-4 & 1 \\ 0 & -2 & x-1 \end{vmatrix}$

$= (x-3) \begin{vmatrix} x-4 & 1 \\ -2 & x-1 \end{vmatrix} = (x-3)(x^2-5x+4+2)$

$= (x-3)^2(x-2)$

(6) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & y & z \\ 1 & x^2 & y^2 & z^2 \\ 1 & x^3 & y^3 & z^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x-1 & y-1 & z-1 \\ 0 & x^2-x & y^2-y & z^2-z \\ 0 & x^3-x^2 & y^3-y^2 & z^3-z^2 \end{vmatrix} = (x-1)(y-1)(z-1) \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

$= (x-1)(y-1)(z-1) \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix} = (x-1)(y-1)(z-1)(y-x)(z-x) \begin{vmatrix} 1 & 1 \\ y+x & z+x \end{vmatrix}$

$= (x-1)(y-1)(z-1)(y-x)(z-x)(z-y)$

(7) $\begin{vmatrix} 1 & x^2 & ab \\ 1 & a^2 & bx \\ 1 & b^2 & ax \end{vmatrix} = \begin{vmatrix} 1 & x^2 & ab \\ 0 & a^2-x^2 & b(x-a) \\ 0 & b^2-a^2 & x(a-b) \end{vmatrix} = (a-x)(b-a) \begin{vmatrix} a+x & -b \\ b+a & -x \end{vmatrix}$

$= (a-x)(b-a)(-x^2-ax+ab+b^2) = (a-x)(b-a)(b-x)(b+x+a)$

(8) $\begin{vmatrix} b+c & c & b \\ c & a+c & a \\ b & a & a+b \end{vmatrix} = \begin{vmatrix} 0 & -2a & -2a \\ c & a+c & a \\ b & a & b+a \end{vmatrix} = \begin{vmatrix} 0 & -2a & 0 \\ c & a+c & -c \\ b & a & b \end{vmatrix} = 2a \begin{vmatrix} c & -c \\ b & b \end{vmatrix} = 4abc$

(9)
$$\begin{vmatrix} 1 & \cos\alpha & \cos 2\alpha \\ 1 & \cos\beta & \cos 2\beta \\ 1 & \cos\gamma & \cos 2\gamma \end{vmatrix} = \begin{vmatrix} 1 & \cos\alpha & \cos 2\alpha \\ 0 & \cos\beta-\cos\alpha & \cos 2\beta-\cos 2\alpha \\ 0 & \cos\gamma-\cos\alpha & \cos 2\gamma-\cos 2\alpha \end{vmatrix} = \begin{vmatrix} 1 & \cos\alpha & \cos 2\alpha \\ 0 & \cos\beta-\cos\alpha & 2(\cos^2\beta-\cos^2\alpha) \\ 0 & \cos\gamma-\cos\alpha & 2(\cos^2\gamma-\cos^2\alpha) \end{vmatrix}$$
$$= (\cos\beta-\cos\alpha)(\cos\gamma-\cos\alpha)\begin{vmatrix} 1 & 2(\cos\beta+\cos\alpha) \\ 1 & 2(\cos\gamma+\cos\alpha) \end{vmatrix} = 2(\cos\beta-\cos\alpha)(\cos\gamma-\cos\alpha)(\cos\gamma-\cos\beta)$$

(10)
$$\begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & a & b & c \\ 1 & 0 & c & b \\ 1 & c & 0 & a \\ 1 & b & a & 0 \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & a & b & c \\ 0 & -a & c-b & b-c \\ 0 & c-a & -b & a-c \\ 0 & b-a & a-b & -c \end{vmatrix}$$
$$= (a+b+c)\begin{vmatrix} -a & c-b & 0 \\ c-a & -b & a-c-b \\ b-a & a-b & a-c-b \end{vmatrix} = (a+b+c)(a-b-c)\begin{vmatrix} -a & c-b & 0 \\ c-a & -b & 1 \\ b-a & a-b & 1 \end{vmatrix}$$
$$= (a+b+c)(a-b-c)\begin{vmatrix} -a & c-b & 0 \\ c-a & -b & 1 \\ b-c & a & 0 \end{vmatrix} = -(a+b+c)(a-b-c)\begin{vmatrix} -a & c-b \\ b-c & a \end{vmatrix}$$
$$= (a+b+c)(a-b-c)\{a^2-(b-c)^2\} = (a+b+c)(a-b-c)(a+b-c)(a+c-b)$$

(11)
$$\begin{vmatrix} x & 0 & 1 & x \\ 1 & x & x & 0 \\ 0 & x & x & 1 \\ x & 1 & 0 & x \end{vmatrix} = \begin{vmatrix} x & 0 & 1 & x \\ 1 & x & x & 0 \\ 0 & x & x & 1 \\ 0 & 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} x & 0 & 1 & x \\ 1 & 0 & 0 & -1 \\ 0 & 2x & 0 & 1 \\ 0 & 1 & -1 & 0 \end{vmatrix} = x\begin{vmatrix} 0 & 0 & -1 \\ 2x & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} - \begin{vmatrix} 0 & 1 & x \\ 2x & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix}$$
$$= 2x^2 - (-2x^2+1) = 4x^2-1$$

9.4 (1)
$$\begin{vmatrix} 1 & 1 & 1 \\ x & a & b \\ x^2 & a^2 & b^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x & a-x & b-x \\ x^2 & a^2-x^2 & b^2-x^2 \end{vmatrix} = (a-x)(b-x)\begin{vmatrix} 1 & 1 \\ a+x & b+x \end{vmatrix} = (a-x)(b-x)(b-a)$$

∴ $a=b$ のとき 不定 $a\neq b$ のとき $x=a, b$

(2)
$$\begin{vmatrix} 1-x & a & a \\ a & 1-x & a \\ a & a & 1-x \end{vmatrix} = 0 \qquad (1-x)^3+2a^3-3a^2(1-x) = 0$$
$$(1-x-a)^2(1-x+2a) = 0 \qquad x = 1-a,\ 1+2a$$

$$\begin{array}{rrrr|l} 1 & 0 & -3a^2 & 2a^3 & a \\ & a & a^2 & -2a^3 & \\ \hline 1 & a & -2a^2 & 0 & \\ & a & 2a^2 & & \\ \hline 1 & 2a & 0 & & \end{array}$$

9.5 (1)
$$\begin{vmatrix} a+b & b & b & \cdots & b \\ b & a+b & b & \cdots & b \\ b & b & a+b & \cdots & b \\ \vdots & & & & \\ b & b & b & \cdots & a+b \end{vmatrix} = \begin{vmatrix} a+nb & a+nb & \cdots & a+nb \\ b & a+b & \cdots & b \\ b & b & \cdots & b \\ b & b & \cdots & a+b \end{vmatrix}$$
$$= (a+nb)\begin{vmatrix} 1 & 0 & 0 & 0 \\ b & a & 0 & 0 \\ \vdots & & & \\ b & 0 & 0 & a \end{vmatrix} = a^{n-1}(a+nb)$$

(2)
$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ -1 & 0 & 3 & \cdots & n-1 & n \\ -1 & -2 & 0 & \cdots & n-1 & n \\ \vdots & & & & & \\ -1 & -2 & -3 & \cdots & 1-n & 0 \end{vmatrix} = n!\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 1 & \cdots & 1 \\ -1 & -1 & 0 & \cdots & 1 \\ \vdots & & & & \\ -1 & -1 & -1 & \cdots & -1 & 0 \end{vmatrix} = n!\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 1 & & \\ -1 & 0 & 1 & & \\ \vdots & & & & 1 \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix} = n!$$

9.6 $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ a & b-a & c-a & d-a \\ a^2 & b^2-a^2 & c^2-a^2 & d^2-a^2 \\ a^3 & b^3-a^3 & c^3-a^3 & d^3-a^3 \end{vmatrix} = (b-a)(c-a)(d-a)\begin{vmatrix} 1 & 1 & 1 \\ b+a & c+a & d+a \\ b^2+ab+a^2 & c^2+ac+a^2 & d^2+da+a^2 \end{vmatrix}$

$= (b-a)(c-a)(d-a)(c-b)(d-b)\begin{vmatrix} 1 & 1 \\ a+b+c & d+b+a \end{vmatrix}$

$= (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$

(1) a^3b^2cの係数 1

(2) 上記の通り

9.7 $\begin{vmatrix} a+x & a+y & a+z \\ b+x & b+y & b+z \\ c+x & c+y & c+z \end{vmatrix} = \begin{vmatrix} a+x & y-x & z-x \\ b+x & y-x & z-x \\ c+x & y-x & z-x \end{vmatrix} = (y-x)(z-x)\begin{vmatrix} a+x & 1 & 1 \\ b+x & 1 & 1 \\ c+x & 1 & 1 \end{vmatrix} = 0$

9.8 $f(x,y,z) = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = xyz\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix}$

$= xyz(y-x)(z-x)\begin{vmatrix} 1 & 1 \\ y+x & z+x \end{vmatrix} = xyz(y-x)(z-x)(z-y)$

$g(x,y,z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ x^4 & y^4 & z^4 \end{vmatrix} = x^2y^2z^2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = x^2y^2z^2(y-x)(z-x)(z-y)$

$h(x,y,z) = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \end{vmatrix} = xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^3 & y^3-x^3 & z^3-x^3 \end{vmatrix}$

$= xyz(y-x)(z-x)\begin{vmatrix} 1 & 1 \\ y^2+xy+x^2 & z^2+zx+x^2 \end{vmatrix} = xyz(y-x)(z-x)(z-y)(x+y+z)$

∴ $g(x,y,z)$ $h(x,y,z)$ は $f(x,y,z)$ で割り切れる

9.9 (1) $\begin{vmatrix} a+x & -a & 0 & 0 & 0 & 0 \\ 0 & a+x & -a & 0 & 0 & 0 \\ 0 & 0 & a+x & -a & 0 & 0 \\ 0 & 0 & 0 & a+x & -a & 0 \\ 0 & 0 & 0 & 0 & a+x & -a \\ -a & 0 & 0 & 0 & 0 & a+x \end{vmatrix} = (a+x)^6 - a^6 = 0$

(2) $\omega = a(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = \frac{a}{2}(1+\sqrt{3}i)$ とおくとき

$x = a\omega^k - a \quad k = 0, 1, 2, 3, 4, 5$

(3) $a=1$ のとき $x = \omega^k - 1$ ∴ $y = \frac{1}{\omega^k - 1}$

9.10 $\begin{vmatrix} 1 & a & b \\ 1 & b & c \\ 1 & c & d \end{vmatrix} \neq 0 \quad \therefore \begin{vmatrix} 1 & a & b \\ 0 & b-a & c-b \\ 0 & c-a & d-b \end{vmatrix} \neq 0 \quad (b-a)(d-b)-(c-b)(c-a) \neq 0$

$bd+ca+cb-b^2-ad-c^2 \neq 0$

$$\begin{vmatrix} 1 & a & b & c+d \\ 1 & b & c & d+a \\ 1 & c & d & a+b \\ 1 & d & a & x \end{vmatrix} = \begin{vmatrix} 1 & a & b & c+d \\ 0 & b-a & c-b & a-c \\ 0 & c-b & d-c & b-d \\ 0 & d-c & a-d & x-a-b \end{vmatrix} = \begin{vmatrix} b-a & c-b & a-c \\ c-b & d-c & b-d \\ d-c & a-d & x-a-b \end{vmatrix} = 0$$

$$\begin{vmatrix} 0 & c-b & a-c \\ 0 & d-c & b-d \\ x-b-c & a-d & x-a-b \end{vmatrix} = 0 \qquad (x-b-c)\begin{vmatrix} c-b & a-c \\ d-c & b-d \end{vmatrix} = 0$$

$$(x-b-c)(cb+bd+ac-b^2-ad-c^2)=0$$

$\therefore\ x=b+c$

9.11 (1) $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ とおくと $A^t=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ $A^{-1}=\frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$|A+A^t|+|A-A^t| = \begin{vmatrix} 2a & b+c \\ b+c & 2d \end{vmatrix} + \begin{vmatrix} 0 & b-c \\ c-b & 0 \end{vmatrix} = 4ad-(b+c)^2+(b-c)^2 = 4|A|$$

$$|A+A^{-1}|+|A-A^{-1}| = \begin{vmatrix} a+\frac{d}{|A|} & b-\frac{b}{|A|} \\ c-\frac{c}{|A|} & d+\frac{a}{|A|} \end{vmatrix} + \begin{vmatrix} a-\frac{d}{|A|} & b+\frac{b}{|A|} \\ c+\frac{c}{|A|} & d-\frac{a}{|A|} \end{vmatrix}$$

$$= \left(a+\frac{d}{|A|}\right)\left(d+\frac{a}{|A|}\right) - bc\left(1-\frac{1}{|A|}\right)^2 + \left(a-\frac{d}{|A|}\right)\left(d-\frac{a}{|A|}\right) - bc\left(1+\frac{1}{|A|}\right)^2$$

$$= ad+\frac{d^2+a^2}{|A|}+\frac{ad}{|A|^2} - 2bc\left(1+\frac{1}{|A|^2}\right) + ad - \frac{d^2+a^2}{|A|} + \frac{ad}{|A|^2}$$

$$= 2ad\left(1+\frac{1}{|A|^2}\right) - 2bc\left(1+\frac{1}{|A|^2}\right) = 2|A|\left(1+\frac{1}{|A|^2}\right) = 2\left(|A|+\frac{1}{|A|}\right)$$

(2) $B=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ とおくと $|B-\lambda E|=0$ より $(a-\lambda)(d-\lambda)-bc=0$

$\lambda^2-(a+d)\lambda+ad-bc=0$ $\lambda=1,2$ とすると $a+d=3,\ ad-bc=2$

$$|B^2-3B| = |B||B-3E| = 2\begin{vmatrix} a-3 & b \\ c & d-3 \end{vmatrix} = 2\{9-(a+d)3+ad-bc\} = 4$$

9.12 4次の行列 $a_{11}\,a_{22}\,a_{33}\,a_{44}$ の係数 $\delta(1\ 2\ 3\ 4)=1$ 偶置換 1

$a_{14}\,a_{21}\,a_{32}\,a_{43}$ $\delta(4\ 1\ 2\ 3)=-1$ 奇置換 -1

9.13 $A=\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ $B=\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ $E=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(1) $AB-BA=\begin{pmatrix} -3 & 2 \\ -1 & 3 \end{pmatrix}$

(2) $\begin{vmatrix} -\lambda & 2 \\ 1 & -\lambda \end{vmatrix}=0$ $\lambda^2-2=0$ $\lambda=\pm\sqrt{2}$

9.14 $A=\begin{pmatrix} a & b \\ b & a \end{pmatrix}=\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$ $B=\begin{pmatrix} b & a \\ a & b \end{pmatrix}=\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ $a=1,\ b=-2$ より

$$|AB| = \begin{vmatrix} -4 & 5 \\ 5 & -4 \end{vmatrix} = -9$$

§ 10 行列式の応用

10.1

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0 \quad A = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad B = \begin{vmatrix} x_1^2+y_1^2 & y_1 & 1 \\ x_2^2+y_2^2 & y_2 & 1 \\ x_3^2+y_3^2 & y_3 & 1 \end{vmatrix}$$

$$C = \begin{vmatrix} x_1^2+y_1^2 & x_1 & 1 \\ x_2^2+y_2^2 & x_2 & 1 \\ x_3^2+y_3^2 & x_3 & 1 \end{vmatrix} \quad D = \begin{vmatrix} x_1^2+y_1^2 & x_1 & y_1 \\ x_2^2+y_2^2 & x_2 & y_2 \\ x_3^2+y_3^2 & x_3 & y_3 \end{vmatrix}$$ とおくと

$A(x^2+y^2) - Bx + Cy - D = 0$ 　$A \neq 0$ のとき

$\therefore \left(x - \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \frac{D}{A} + \frac{1}{4A^2}(B^2+C^2)$

$\therefore$ 中心 $\left(\frac{B}{2A}, \frac{-C}{2A}\right)$, 半径 $\frac{1}{2A}\sqrt{4AD+B^2+C^2}$ の円

10.2 (1) $\begin{vmatrix} y & y_1 & y_2 \\ x & x_1 & x_2 \\ 1 & 1 & 1 \end{vmatrix} = 0$ 　$y(x_1-x_2) - x(y_1-y_2) + y_1x_2 - x_1y_2 = 0$

$\therefore (x_1, y_1), (x_2, y_2)$ を通る直線

(2) $\begin{vmatrix} y & y_1 & y_2 & y_3 \\ x^2 & x_1^2 & x_2^2 & x_3^2 \\ x & x_1 & x_2 & x_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$ は 3点 $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ を通る放物線

10.3 $a_ix + b_iy + c_i = 0$ 　$i = 1, 2, 3$ は 3直線で $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$ のとき

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = A$ とおくと

$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$ の交点は $x = \frac{-\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{A}$, $y = \frac{-\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{A}$

これが $a_3x + b_3y + c_3 = 0$ 上にある.

$a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$

$\therefore \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

10.4 2点 $(x_1, y_1), (x_2, y_2)$ を通る直線の方程式は

$\begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} = 0$ 　これが (x_3, y_3) を通る $\begin{vmatrix} 1 & x_3 & y_3 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} = 0$

10.5　$O(0,0)$, $A_1(a_{11}, a_{12})$, $A_2(a_{21}, a_{22})$ とするとき

OA_1, OA_2 を二辺とする平行四辺形の面積は

$|\vec{OA_1} \times \vec{OA_2}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ の絶対値

10.6　$\vec{A} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$, $\vec{B} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$,

$\vec{C} = c_x\vec{i} + c_y\vec{j} + c_z\vec{k}$ が同一平面内にあるとき

$\alpha\vec{A} + \beta\vec{B} + \gamma\vec{C} = 0 \quad (\alpha, \beta, \gamma) \neq (0, 0, 0)$

$\therefore$ $a_x\alpha + a_y\beta + a_z\gamma = 0$

$b_x\alpha + b_y\beta + b_z\gamma = 0$

$c_x\alpha + c_y\beta + c_z\gamma = 0$

が原点以外に解をもつ

$$\therefore \begin{vmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{vmatrix} = 0$$

§11 階数と方程式

.1 (1)
$$\begin{cases} x_1+2x_2-x_3+3x_4-2x_5=1 \\ 2x_1+4x_2+x_3+3x_4-3x_5=2 \\ -x_1-2x_2+2x_3-4x_4-x_5=1 \\ 3x_1+6x_2+6x_4-5x_5=4 \end{cases}$$

$$\mathrm{Rank}\begin{pmatrix} 1 & 2 & -1 & 3 & -2 \\ 2 & 4 & 1 & 3 & -3 \\ -1 & -2 & 2 & -4 & -1 \\ 3 & 6 & 0 & 6 & -5 \end{pmatrix} = \mathrm{Rank}\begin{pmatrix} 1 & 2 & -1 & 3 & -2 \\ 0 & 0 & 3 & -3 & 1 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 3 & -3 & 1 \end{pmatrix} = \mathrm{Rank}\begin{pmatrix} 1 & 2 & -1 & 3 & -2 \\ 0 & 0 & 3 & -3 & 1 \\ 0 & 0 & 1 & -1 & -3 \end{pmatrix} = 3$$

$$\mathrm{Rank}\begin{pmatrix} 1 & 2 & -1 & 3 & -2 & 1 \\ 2 & 4 & 1 & 3 & -3 & 2 \\ -1 & -2 & 2 & -4 & -1 & 1 \\ 3 & 6 & 0 & 6 & -5 & 4 \end{pmatrix} = \mathrm{Rank}\begin{pmatrix} 1 & 2 & -1 & 3 & -2 & 1 \\ 0 & 0 & 3 & -3 & 1 & 0 \\ 0 & 0 & 1 & -1 & -3 & 2 \\ 0 & 0 & 3 & -3 & 1 & 1 \end{pmatrix} = \mathrm{Rank}\begin{pmatrix} 1 & 2 & -1 & 3 & -2 & 1 \\ 0 & 0 & 3 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{10}{3} & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$= 4$

∴ 不能

(2)
$$\begin{cases} (m+1)x_1+x_2+x_3=m-2 \\ x_1+(m+1)x_2+x_3=-2 \\ x_1+x_2+(m+1)x_3=-2 \end{cases}$$

$(m+3)(x_1+x_2+x_3)=m-6$

i) $m+3\neq 0$ のとき

$x_1+x_2+x_3=\dfrac{m-6}{m+3}$

$x_1=\dfrac{m}{m+3}$　　$x_2=x_3=\dfrac{-3}{m+3}$

ii) $m+3=0$ のとき 不能

iii) $m=0$ のとき　$x_1+x_2+x_3=-2$

(3)
$$\begin{cases} x+2y+3z+4u=2 \\ 2x-y+z-u=-4 \\ 3x+y+4z+3u=-2 \\ x+3y+4z+5u=2 \end{cases}$$

x	y	z	u	
1	2	3	4	2
2	−1	1	−1	−4
3	1	4	3	−2
1	3	4	5	2
1	2	3	4	2
0	−5	−5	−9	−8
0	−5	−5	−9	−8
0	1	1	1	0
1	0	1	2	2
0	0	0	−4	−8
0	0	0	−4	−8
0	1	1	1	0
1	0	1	0	−2
0	0	0	1	2
0	0	0	1	2
0	1	1	0	−2

∴ $x+z=-2$
$y+z=-2$
$u=2$

$$\begin{cases} x=y=k \\ z=-2-k \\ u=2 \end{cases}$$　kは任意

(4) $\begin{cases} x+y+z=1 \\ ax+by+cz=d \\ a^2x+b^2y+c^2z=d^2 \end{cases}$

x	y	z	
1	1	1	1
a	b	c	d
a^2	b^2	c^2	d^2
1	1	1	1
0	$b-a$	$c-a$	$d-a$
0	b^2-a^2	c^2-a^2	d^2-a^2
1	1	1	1
0	1	$\frac{c-a}{b-a}$	$\frac{d-a}{b-a}$
0	0	$(c-b)(c-a)$	$(d-a)(d-b)$

$x+y+z=1$

$y+\frac{c-a}{b-a}z=\frac{d-1}{b-a}$

$z=\frac{(d-b)(d-a)}{(c-b)(c-a)}$

i) a, b, c のどの2つも異なるとき,

$x=\frac{(d-b)(c-d)}{(a-b)(c-a)}$ $y=\frac{(d-a)(d-c)}{(b-a)(b-c)}$ $z=\frac{(d-b)(d-a)}{(c-b)(c-a)}$

ii) $a=b=c=1$ のとき

$x+y+z=1$

iii) その他 不能

(5) $\begin{cases} 4x+2y+3z+w=1 \\ x+y-2w=2 \\ 5x+2y+4z-w=0 \end{cases}$

x	y	z	w	
4	2	3	1	1
1	1	0	−2	2
5	2	4	−1	0
0	−2	3	9	−7
1	1	0	−2	2
0	−3	4	9	−10
0	−2	3	9	−7
1	1	0	−2	2
0	−1	1	0	−3
0	0	1	9	−1
1	1	0	−2	2
0	−1	1	0	−3

$z+9w=-1$

$x+y-2w=2$

$-y+z=-3$

$z=k$ $y=k+3$ $w=-\frac{k+1}{9}$

$x=2+2w-y=2-\frac{2k+2}{9}-k-3$

$=-\frac{9(k+1)+2(k+1)}{9}=\frac{-11(k+1)}{9}$

∴ $x=-\frac{11(k+1)}{9}$ $y=k+3$ $z=k$ $w=-\frac{k+1}{9}$

(6) $\begin{cases} ax+y+z=1 \\ x+ay+z=a \\ x+y+az=a^2 \end{cases}$

x	y	z	
a	1	1	1
1	a	1	a
1	1	a	a^2
0	$1-a$	$1-a^2$	$1-a^3$
0	$a-1$	$1-a$	$a-a^2$
1	1	a	a^2
0	0	$2-a-a^2$	$1+a-a^2-a^3$
0	$a-1$	$1-a$	$a-a^2$
1	1	a	a^2

$-(a+2)(a-1)z=(1+a)^2(1-a)$

$(a-1)y+(1-a)z=a(1-a)$

$x+y+az=a^2$

i) $a\neq1$ $a\neq-2$ のとき

$z=\frac{(a+1)^2}{a+2}$ $y=\frac{1}{a+2}$ $x=-\frac{a+1}{a+2}$

ii) $a=1$ のとき $x+y+z=1$

iii) $a=-2$ のとき 不能

(7)
$$\begin{cases} 2x_1 + 4x_2 + x_3 = 7 \\ 3x_1 + 2x_2 + 3x_3 = 7 \\ 5x_1 - 4x_2 + 4x_3 = 9 \end{cases}$$

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ 2 & 4 & 1 & 7 \\ 3 & 2 & 3 & 7 \\ 5 & -4 & 4 & 9 \\ \hline 2 & 4 & 1 & 7 \\ 1 & -2 & 2 & 6 \\ 0 & -10 & 0 & -5 \\ \hline 2 & 0 & 1 & 5 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & -3 & 3 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & \frac{1}{2} \end{array}$$

$x_2 = \frac{1}{2}$ $x_3 = -1$

$x_1 = 3$

$x_1 = 3$, $x_2 = \frac{1}{2}$, $x_3 = -1$

(8)
$$x_2 + x_3 + x_4 = 2$$
$$3x_1 + 2x_2 + x_3 + 2x_4 = 5$$
$$x_1 + x_2 + x_3 + 2x_4 = 4$$

$$\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 2 \\ 3 & 2 & 1 & 2 & 5 \\ 1 & 1 & 1 & 2 & 4 \\ \hline 0 & 1 & 1 & 1 & 2 \\ 3 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 2 \\ \hline 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & -1 & -3 & -5 \\ 1 & 0 & 0 & 1 & 2 \end{array}$$

$$\begin{cases} x_2 + x_3 + x_4 = 2 \\ x_3 + 3x_4 = 5 \\ x_1 + x_4 = 2 \end{cases}$$

$x_4 = k$ $x_1 = 2-k$ $x_3 = 5-3k$ $x_2 = 2-5+3k-k = -3+2k$

$\therefore x_1 = 2-k$. $x_2 = -3+2k$ $x_3 = 5-3k$ $x_4 = k$

(9)
$$\begin{pmatrix} 3 & 5 & 1 \\ 2 & 3 & -2 \\ 4 & 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix}$$

$x_1 = 1$ $x_2 = 0$ $x_3 = -2$

(10)
$$x - y + z = 1$$
$$-2x - y - 2z = 3$$
$$4x + 3y + 3z = 1$$

$$\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -2 & -1 & -2 & 3 \\ 4 & 3 & 3 & 1 \\ \hline 1 & -1 & 1 & 1 \\ 0 & -3 & 0 & 5 \\ 0 & 7 & -1 & -3 \\ \hline 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -\frac{5}{3} \\ 0 & 7 & -1 & -3 \\ \hline 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -\frac{5}{3} \\ 0 & 0 & -1 & \frac{26}{3} \end{array}$$

$x = 8$

$y = -\frac{5}{3}$

$y = -\frac{26}{3}$

11.2
(1)
$$\begin{cases} x + 2y + 4z = 0 \\ 5x + y + 6z = 0 \\ kx + 3y + 2z = 0 \end{cases}$$

$\begin{vmatrix} 1 & 2 & 4 \\ 5 & 1 & 6 \\ k & 3 & 2 \end{vmatrix} = 0$ より $k = -3$

(2) $k = -3$ のとき 解は $z = t$ とおくと $x = -\frac{8}{9}t$ $z = -\frac{14}{9}t$

$\therefore$ 一次元 $(x, y, z) = t\left(-\frac{8}{9}, -\frac{14}{9}, 1\right)$

11.3 $\begin{cases} mx+y-3z=0 \\ 5x-3y-mz=0 \\ 4x-7y+(m+1)z=0 \end{cases}$ $\begin{vmatrix} m & 1 & -3 \\ 5 & -3 & -m \\ 4 & -7 & m+1 \end{vmatrix}=0$ より $m=2,\ -\frac{16}{5}$

11.4 $\begin{cases} a_{11}x_1+a_{12}x_2+a_{13}x_3=b_1 & \cdots ① \\ a_{21}x_1+a_{22}x_2+a_{23}x_3=b_2 & \cdots ② \\ a_{31}x_1+a_{32}x_2+a_{33}x_3=b_3 & \cdots ③ \end{cases}$ とおくとき

行列式 $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ で a_{ij} の係数を A_{ij} とおくと

$\sum_{i=1}^{3} a_{ij}A_{ij}=\sum_{j=1}^{3} a_{ij}A_{ij}=|A|$ $\quad \sum_{j} a_{ij}A_{kj}=\sum_{i} a_{ij}A_{ik}=0$ $\quad (i\neq k,\ j\neq k)$

∴ $①A_{11}+②A_{21}+③A_{31}=b_1A_{11}+b_2A_{21}+b_3A_{31}$

∴ $|A|x_1=\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$

∴ $x_1=\frac{1}{|A|}\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$ $\quad x_2=\frac{1}{|A|}\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$ $\quad x_3=\frac{1}{|A|}\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$

11.5 (1) $\begin{cases} a_{11}x_1+a_{12}x_2=a_1 \\ a_{21}x_1+a_{22}x_2=a_2 \end{cases}$ $\quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\neq 0$ のとき

$x_1=\frac{1}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}\begin{vmatrix} a_1 & a_{12} \\ a_2 & a_{22} \end{vmatrix}$ $\quad x_2=\frac{1}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}\begin{vmatrix} a_{11} & a_1 \\ a_{21} & a_2 \end{vmatrix}$

(2) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=0$ のとき $a_{11}a_{22}-a_{12}a_{21}=0$

$\frac{a_{11}}{a_{21}}=\frac{a_{12}}{a_{22}}=\lambda$ とおくと $(a_{11},a_{12})=\lambda(a_{21},a_{22})$

$a_{11}x_1+a_{12}x_2=a_1$ ∴ $\lambda(a_{21}x_1+a_{22}x_2)=a_1$

また $\lambda(a_{21}x_1+a_{22}x_2)=\lambda a_2$ ∴ $a_1=\lambda a_2$

∴ $\frac{a_{11}}{a_{21}}=\frac{a_{12}}{a_{22}}=\frac{a_1}{a_2}$

11.6 $\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & \alpha \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & \alpha & 0 \\ \hline 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & \alpha-1 & 1 \end{array}\right)$$

$x_1 - x_2 = 1$

$x_2 - x_3 = 1$

$x_3 - x_4 = 1$

$(\alpha-1)x_4 = 1$

$\alpha \neq 1$ のとき

$x_4 = \frac{1}{\alpha-1}$ $x_3 = \frac{\alpha}{\alpha-1}$ $x_2 = \frac{2\alpha-1}{\alpha-1}$ $x_1 = \frac{3\alpha-2}{\alpha-1}$

$\alpha = 1$ のとき 不能

11.7 $M = \begin{pmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{pmatrix}$ $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ $Y = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

(1) $|M| = -(a+b+c)(b+c-a)(a+b-c)(c+a-b)$

(2) $MX = \begin{pmatrix} ax_2 + bx_3 + cx_4 \\ ax_1 + cx_3 + bx_4 \\ bx_1 + cx_2 + ax_4 \\ cx_1 + bx_2 + ax_3 \end{pmatrix}$

(3) $|M|x_1 = \begin{vmatrix} 1 & a & b & c \\ 1 & 0 & c & b \\ 0 & c & 0 & a \\ 0 & b & a & 0 \end{vmatrix} = a(a-b+c)(a+b-c)$

$|M|x_2 = \begin{vmatrix} 0 & 1 & b & c \\ a & 1 & c & b \\ b & 0 & 0 & a \\ c & 0 & a & 0 \end{vmatrix} = a(a-b+c)(a+b-c)$

$|M|x_3 = \begin{vmatrix} 0 & a & 1 & c \\ a & 0 & 1 & b \\ b & c & 0 & a \\ c & b & 0 & 0 \end{vmatrix} = (b+c)(b-c-a)(a+b-c)$

$|M|x_4 = \begin{vmatrix} 0 & a & b & 1 \\ a & 0 & c & 1 \\ b & c & 0 & 0 \\ c & b & a & 0 \end{vmatrix} = (b+c)(b-c+a)(b-c-a)$

(4) $|M| = 0$

$(a+b+c)(b+c-a)(a+c-b)(a+b-c) = 0$

11.8 $\mathrm{Rank}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 5+x \\ 7 & 5+x & 9 & 1+3x \end{pmatrix} = \mathrm{Rank}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 5 & -4 & -8 & x-15 \\ 7 & x-9 & -12 & 3x-27 \end{pmatrix}$

$= \mathrm{Rank}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & -8 & x-15 \\ 0 & x-9 & -12 & 3(x-9) \end{pmatrix} = 2$

$\therefore \frac{x-9}{4} = \frac{-3}{2} = \frac{-3(x-9)}{x-15}$ $x = 3$

11.9

$$\begin{cases} 2x+3y+z=-4 \\ 4x+y-3z=2 \\ -x+2y+2z=-6 \end{cases}$$

$$\begin{vmatrix} 2 & 3 & 1 \\ 4 & 1 & -3 \\ -1 & 2 & 2 \end{vmatrix} = 10, \quad \begin{vmatrix} -4 & 3 & 1 \\ 2 & 1 & -3 \\ -6 & 2 & 2 \end{vmatrix} = 20$$

$$\begin{vmatrix} 2 & -4 & 1 \\ 4 & 2 & -3 \\ -1 & -6 & 2 \end{vmatrix} = -30 \quad \begin{vmatrix} 2 & 3 & -4 \\ 4 & 1 & 2 \\ -1 & 2 & -6 \end{vmatrix} = 10$$

$\therefore x=2,\ y=-3,\ z=1$

11.10 (1)

$$A\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\2\\1\end{pmatrix} \quad A\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2\\4\\4\\2\end{pmatrix} \quad A\begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}4\\3\\8\\-1\end{pmatrix}$$

$$a\begin{pmatrix}1\\2\\2\\1\end{pmatrix} + b\begin{pmatrix}2\\4\\4\\2\end{pmatrix} + c\begin{pmatrix}4\\3\\8\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\2\\-1\end{pmatrix}$$

$$\begin{cases} a+2b+4c=1 \\ 2a+4b+3c=0 \\ 2a+4b+8c=2 \\ a+2b-c=-1 \end{cases} \qquad \begin{aligned} a+2b+4c&=1 \\ -5c&=-2 \\ 5c&=2 \\ -5c&=-2 \end{aligned}$$

$c=\frac{2}{5}$ 　 $a+2b=-\frac{3}{5}$ 　 $b=0$ とおくと

(a, b, c) 　 $\left(-\frac{3}{5}, 0, \frac{2}{5}\right)$

$$A\left\{\frac{-3}{5}\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{2}{5}\begin{pmatrix}1\\-1\\2\end{pmatrix}\right\} = A\begin{pmatrix}-\frac{1}{5}\\-1\\\frac{1}{5}\end{pmatrix} = \begin{pmatrix}1\\0\\2\\-1\end{pmatrix}$$

$$\therefore X = \begin{pmatrix}-\frac{1}{5}\\-1\\\frac{1}{5}\end{pmatrix}$$

(2) $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\-1\\2\end{pmatrix}$ は一次独立

$\begin{pmatrix}1\\2\\2\\1\end{pmatrix}, \begin{pmatrix}2\\4\\4\\2\end{pmatrix}, \begin{pmatrix}4\\3\\8\\-1\end{pmatrix}$ は一次従ぞくで $\begin{pmatrix}1\\2\\2\\1\end{pmatrix}, \begin{pmatrix}4\\3\\8\\-1\end{pmatrix}$ は一次独立

$\therefore \mathrm{Rank}\ A = 2$

§12 固有値

12.1 $A=\begin{pmatrix}0&1&0\\0&0&1\\-c&-b&-a\end{pmatrix}$ $f(x)=|xE-A|=\begin{vmatrix}x&-1&0\\0&x&-1\\c&b&x+a\end{vmatrix}$

$$=x^2(x+a)+c+bx=x^3+ax^2+bx+c$$

$\therefore f(1)=1+a+b+c=-18$ $a+b+c=-19$

$f(2)=8+4a+2b+c=-24$ $4a+2b+c=-32$

$f(3)=27+9a+3b+c=-20$ $9a+3b+c=-47$

1	1	1	−19
4	2	1	−32
9	3	1	−47
1	1	1	−19
3	1	0	−13
5	1	0	−15
1	1	1	−19
3	1	0	−13
2	0	0	−2
0	0	1	−8
0	1	0	−10
1	0	0	−1

$\therefore a=-1 \quad b=-10 \quad c=-8$

$\therefore x^3-x^2-10x-8=0$

1	−1	−10	−8	−1
	−1	2	8	
1	−2	−8	0	
	−2	8		−2
1	−4	0		

$(x+1)(x+2)(x-4)=0$

$\therefore x=-1,\ -2,\ 4$

12.2 $\begin{pmatrix}x\\y\\z\end{pmatrix}\to\begin{pmatrix}x+y\\y+z\\z+x\end{pmatrix}=\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$

$A=\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix}$ $|A-\lambda E|=0$

$\begin{vmatrix}1-\lambda&1&0\\0&1-\lambda&1\\1&0&1-\lambda\end{vmatrix}=0$ $(1-\lambda)^3+1=0$ $1-3\lambda+3\lambda^2-\lambda^3+1=0$

$\lambda^3-3\lambda^2+3\lambda-2=0$

1	−3	3	−2	2
	2	−2	2	
1	−1	1	0	

$(\lambda-2)(\lambda^2-\lambda+1)=0$

$\therefore \lambda=2,\ \frac{1}{2}(1\pm\sqrt{3}i)$

12.3 $A=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$ $|A-xE|=0$ $-x^3+2+3x=0$

$x^3-3x-2=0$

$(x+1)^2(x-2)=0$

1	0	−3	−2	−1
	−1	1	2	
1	−1	−2	0	
	−1	2		
1	−2	0		

$x=-1,\ 2$

12.4 $\bar{A}^t=A^{-1}$ のとき Aをユニタリ行列という $\bar{A}$は, $A=(a_{ij})$のとき $\bar{A}=(\bar{a}_{ij})$

$\bar{a}_{ij}$はa_{ij}の共役複素数

Aの固有値をλ, その固有ベクトルをxとおくと

$Ax = \lambda x$ $\quad\quad \bar{x}^t x = |x|^2$

$\therefore \bar{\lambda} \bar{x}^t \cdot \lambda x = \bar{x}^t \bar{A}^t A x = \bar{x}^t x$

$\therefore |\lambda|^2 |x|^2 = |x|^2 \quad \therefore \lambda^2 = 1$

12.5 $\quad A = \begin{pmatrix} 0 & 1 & a \\ 1 & -a & 0 \\ a & 0 & 1 \end{pmatrix}$

(1) $\text{Rank}\, A = \text{Rank} \begin{pmatrix} 0 & 1 & a \\ 1 & 0 & a^2 \\ 0 & a^2 & 1 \end{pmatrix} = \text{Rank} \begin{pmatrix} 0 & 1 & a \\ 1 & 0 & a^2 \\ 0 & 0 & 1-a^3 \end{pmatrix}$

$1 - a^3 = (1-a)(1+a+a^2)$

$\therefore \text{Rank}\, A = 3 \quad (a \neq 1). \quad \text{Ran}\, A = 2 \quad (a = 1)$

(2) $\quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ とおくと

$A\vec{x} = \begin{pmatrix} 0 & 1 & a \\ 1 & -a & 0 \\ a & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$a \neq 1$ とき $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$a = 1$ のとき $\quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad x = y = -z$

$\therefore \quad \vec{x} = t \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

12.6
(1) $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad\quad A = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}$

A の固有値 $\quad (6-\lambda)(3-\lambda) - 4 = 0 \quad \lambda^2 - 9\lambda + 14 = 0$

$(\lambda - 2)(\lambda - 7) = 0 \quad \lambda = 2, 7$

$\lambda = 2$ のとき

$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad 2p + q = 0 \quad \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda = 7 \quad \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad -p + 2q = 0 \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(2) $\quad \vec{e}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \vec{e}_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$ とおくと

$\vec{e}_1 \cdot \vec{e}_2 = 0 \quad |\vec{e}_1| = |\vec{e}_2| = 1$

12.7 $A=\begin{pmatrix} a & b & c \\ b & 1 & 0 \\ c & 0 & 1 \end{pmatrix}$ の固有値がすべて正

$|A-\lambda E|=0$ $\begin{vmatrix} a-\lambda & b & c \\ b & 1-\lambda & 0 \\ c & 0 & 1-\lambda \end{vmatrix}=0$ $(1-\lambda)^2(a-\lambda)-c^2(1-\lambda)-b^2(1-\lambda)=0$

$(1-\lambda)\{(1-\lambda)(a-\lambda)-b^2-c^2\}=0$

$\therefore \lambda=1.$ $\lambda^2-(a+1)\lambda+a-b^2-c^2=0$ の二根が正

$\therefore a+1>0$ $a>b^2+c^2$

$|A|=\begin{vmatrix} a & b & c \\ b & 1 & 0 \\ c & 0 & 1 \end{vmatrix}=a-b^2-c^2>0$

$\therefore$ 固有値がすべて正ならば $|A|>0$ 逆に $|A|>0$ ならば

$a>b^2+c^2$ $a+1>0$ $\therefore$ 固有値はすべて正

12.8 3次の直交行列 $A=(a_{ij})$ $|A|=1$

(1) $A^tA=E$ $A^{-1}=A^t$

$\therefore \begin{pmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$

$\therefore a_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ $a_{22}=\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$ $a_{33}=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

(2) $|A-\lambda E|=0$

$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}=0$

$-\lambda^3+(a_{11}+a_{22}+a_{33})\lambda^2-\left(\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right)\lambda+|A|=0$

$\therefore \lambda^3-tr(A)\lambda^2+tr(A)\lambda+|A|=0$

$\lambda^3-1-tr(A)\lambda(\lambda-1)=0$ $(\lambda-1)\{\lambda^2+(1-tr(A))\lambda+1\}=0$

$\therefore \lambda=1$ は固有値 $tr(A)=-1$ のとき $\lambda^2+2\lambda+1=0$

$\therefore \lambda=-1$ は固有値.

12.9 $A=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}5&6\\7&8\end{pmatrix}-\lambda\begin{pmatrix}1&0\\0&1\end{pmatrix}$

(1) $A=\begin{pmatrix}19-\lambda & 22\\ 43 & 50-\lambda\end{pmatrix}$

(2) $|A|=0 \quad (19-\lambda)(50-\lambda)-22\times 43=0$

$\lambda^2-69\lambda+4=0$

$\therefore \lambda=\dfrac{69\pm\sqrt{4745}}{2}$

12.10 A 3次の行列 B 対称 C 交代行列

$A=B+C \quad {}^tB=B \quad {}^tC=-C$

$\therefore A+{}^tA=2B \qquad A-{}^tA=2C$

$B=\frac{1}{2}(A+{}^tA) \qquad C=\frac{1}{2}(A-{}^tA)$

(1) $B=\frac{1}{2}\left\{\begin{pmatrix}0&0&3\\-2&0&2\\-1&0&1\end{pmatrix}+\begin{pmatrix}0&-2&-1\\0&0&0\\3&2&1\end{pmatrix}\right\}=\begin{pmatrix}0&-1&1\\-1&0&1\\1&1&1\end{pmatrix}$

$C=\frac{1}{2}\left\{\begin{pmatrix}0&0&3\\-2&0&2\\-1&0&1\end{pmatrix}-\begin{pmatrix}0&-2&-1\\0&0&0\\3&2&1\end{pmatrix}\right\}=\begin{pmatrix}0&1&2\\-1&0&1\\-2&-1&0\end{pmatrix}$

$3b_0=b_{11}+b_{22}+b_{33}=1 \quad \therefore b_0=\frac{1}{3}$

$b_{11}'=b_{11}-b_0=-\frac{1}{3} \quad b_{22}'=b_{22}-b_0=-\frac{1}{3} \quad b_{33}'=b_{33}-b_0=\frac{2}{3}$

$\therefore B'=\begin{pmatrix}-\frac{1}{3}&-1&1\\-1&-\frac{1}{3}&1\\1&1&\frac{2}{3}\end{pmatrix}$

$|B-\lambda E|=0 \quad \begin{vmatrix}-\lambda&-1&1\\-1&-\lambda&1\\1&1&1-\lambda\end{vmatrix}=0 \quad -\lambda^3+\lambda^2+3\lambda-3=0$

$\lambda_1=1 \quad \lambda_2=\sqrt{3} \quad \lambda_3=-\sqrt{3}$

$J=-(\frac{1}{9}-\frac{2}{9}-\frac{2}{9})+3=\frac{10}{3}$

(2) $|B'-\lambda E|=0 \quad \begin{vmatrix}-\frac{1}{3}-\lambda&-1&1\\-1&-\frac{1}{3}-\lambda&1\\1&1&\frac{2}{3}-\lambda\end{vmatrix}=0$

$-\lambda^3+J\lambda+|B'|=0$

$\therefore J=-(\lambda_1'\lambda_2'+\lambda_2'\lambda_3'+\lambda_3'\lambda_1')$

(3) $J=-(b_{11}'b_{22}'+b_{22}'b_{33}'+b_{33}'b_{11}')+b_{12}^2+b_{13}^2+b_{23}^2$

$=-\{b_{11}b_{22}+b_{22}b_{33}+b_{33}b_{11}-2b_0(b_{11}+b_{22}+b_{33})+3b_0^2\}+b_{12}^2+b_{13}^2+b_{23}^2$

$= -\{b_{11}b_{22}+b_{22}b_{33}+b_{33}b_{11}-3b_0^2\}+b_{12}^2+b_{13}^2+b_{23}^2$

$= -(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)+3b_0^2 \qquad 3b_0=\lambda_1+\lambda_2+\lambda_3$

$= -(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)+2b_0(\lambda_1+\lambda_2+\lambda_3)-3b_0^2$

$= -\{(\lambda_1-b_0)(\lambda_2-b_0)+(\lambda_2-b_0)(\lambda_3-b_0)+(\lambda_3-b_0)(\lambda_1-b_0)\}$

12.11 $A(t)=\begin{pmatrix} t & 1-t \\ 1-t & t \end{pmatrix} \qquad |A(t)|=\begin{vmatrix} t & 1-t \\ 1-t & t \end{vmatrix}=t^2-(1-t)^2=2t-1$

(1) $t\neq\frac{1}{2}$ のとき Rank $A(t)=2$

$t=\frac{1}{2}$ のとき Rank $A(t)=1$

(2) $A(t_1)\,A(t_2)=\begin{pmatrix} t_1 & 1-t_1 \\ 1-t_1 & t_1 \end{pmatrix}\begin{pmatrix} t_2 & 1-t_2 \\ 1-t_2 & t_2 \end{pmatrix}$

$=\begin{pmatrix} 1-t_1-t_2+2t_1t_2 & t_1+t_2-2t_1t_2 \\ t_1+t_2-2t_1t_2 & 1-t_1-t_2+2t_1t_2 \end{pmatrix}=\begin{pmatrix} \tau & 1-\tau \\ 1-\tau & \tau \end{pmatrix}$

$\tau=1-(t_1+t_2)+2t_1t_2$

(3) $\begin{vmatrix} t-\lambda & 1-t \\ 1-t & t-\lambda \end{vmatrix}=0 \qquad (t-\lambda)^2-(1-t)^2=0$

$(t-\lambda-1+t)(t-\lambda+1-t)=0$

$\lambda=1,\ 2t-1$

12.12 $A=\begin{pmatrix} 2 & 3 & -2 \\ -2 & -2 & 1 \\ 4 & -1 & 6 \end{pmatrix} \qquad |A-\lambda E|=0$

$\begin{vmatrix} 2-\lambda & 3 & -2 \\ -2 & -2-\lambda & 1 \\ 4 & -1 & 6-\lambda \end{vmatrix}=0$ より $\lambda^3-6\lambda^2+11\lambda-6=0$

$(\lambda-1)(\lambda-2)(\lambda-3)=0$

$\therefore \lambda=1,\ 2,\ 3$

最大値 3

$\begin{pmatrix} -1 & 3 & -2 \\ -2 & -5 & 1 \\ 4 & -1 & 3 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} -p+3q-2r=0 \\ -2p-5q+r=0 \\ 4p-q+3r=0 \end{matrix}$

$-5p-7q=0 \qquad q=-\frac{5}{7}p \qquad r=2p+5q=(2-\frac{25}{7})p=-\frac{11}{7}$

$\therefore \begin{pmatrix} p \\ q \\ r \end{pmatrix}=\begin{pmatrix} 7 \\ -5 \\ -11 \end{pmatrix}$

§.13 固有ベクトル

13.1 (1) $\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 4-\lambda & 1 \\ -1 & 1 & 2-\lambda \end{vmatrix} = 0$ $\qquad (2-\lambda)^2(4-\lambda)+2-(4-\lambda)-2(2-\lambda)=0$

$-(\lambda-2)(\lambda-1)(\lambda-5)=0$

$\lambda = 1, 2, 5$

$\lambda=1$ $\begin{pmatrix} 1 & -1 & -1 \\ -1 & 3 & 1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad p-q-r=0,\ -p+3q+r=0 \quad q=0 \quad \therefore \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda=2$ $\begin{pmatrix} 0 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad q+r=0,\ -p+2q+r=0,\ -p+q=0 \quad p=q=-r \quad \therefore \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$\lambda=5$ $\begin{pmatrix} -3 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -3 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad -3p+q+r=0,\ -p-q+r=0,\ -p+q-3r=0 \quad p=-r,\ q=2r \quad \therefore \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

(2) $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$ $\qquad (2-\lambda)^3 - 3(2-\lambda)+2=0$

$(1-\lambda)^2(4-\lambda)=0$

$\lambda = 1, 4$

$\lambda=1$ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad p+q+r=0 \quad \therefore \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda=4$ $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad -2p+q+r=0,\ p-2q+r=0,\ p+q-2r=0 \quad p=q=r \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(3) $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \quad (2-\lambda)^2-1=0 \quad (1-\lambda)(3-\lambda)=0$

$\lambda = 1, 3$

$\lambda=1$ $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad p+q=0 \quad \therefore \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda=3$ $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad -p+q=0 \quad \therefore \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(4) $\begin{vmatrix} -1-\lambda & 8 \\ -2 & 7-\lambda \end{vmatrix} = 0 \qquad -(1+\lambda)(7-\lambda)+16=0$

$(\lambda-3)^2=0$

$\lambda=3$

$\begin{pmatrix} -4 & 8 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad -p+2q=0 \quad \therefore \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(5) $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \qquad \lambda^2 - 1 = 0$

$\lambda = -1, 1$

$\lambda = -1 \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad p + q = 0 \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = 1 \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad p - q = 0 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(6) $\begin{vmatrix} a-\lambda & 1 & 0 \\ 1 & a-\lambda & 1 \\ 0 & 1 & a-\lambda \end{vmatrix} = 0 \quad (a-\lambda)^3 - 2(a-\lambda) = 0$

$(a-\lambda)(a-\lambda+\sqrt{2})(a-\lambda-\sqrt{2}) = 0$

$\lambda = a, \ a+\sqrt{2}, \ a-\sqrt{2}$

$\lambda = a \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} q = 0 \\ p + r = 0 \end{matrix} \quad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda = a+\sqrt{2} \quad \begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} -\sqrt{2}p + q = 0 \\ p - \sqrt{2}q + r = 0 \\ q - \sqrt{2}r = 0 \end{matrix} \quad \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$

$\lambda = a-\sqrt{2} \quad \begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \sqrt{2}p + q = 0 \\ p + \sqrt{2}q + r = 0 \\ q + \sqrt{2}r = 0 \end{matrix} \quad \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$

(7) $\begin{vmatrix} -\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0 \quad \lambda^2(2-\lambda) + 2 - (2-\lambda) - 2\lambda = 0$

$-\lambda(\lambda-1)^2 = 0$

$\lambda = 0, 1$

$\lambda = 0 \quad \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} q + r = 0 \\ p + 2q + r = 0 \\ p + q = 0 \end{matrix} \quad \therefore \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\lambda = 1 \quad \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad p + q + r = 0 \quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

(8) $\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 2 & -1-\lambda \end{vmatrix} = 0 \quad -\lambda^2(1+\lambda) + 2\lambda = 0$

$\lambda(\lambda-1)(\lambda+2) = 0$

$\lambda = 0, 1, -2$

$\lambda=0$ $\quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad q=0 \quad r=0 \quad \therefore \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda=1$ $\quad \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad p=q=r \quad \therefore \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda=-2$ $\quad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad \begin{matrix} 2p+q=0 \\ 2q+r=0 \end{matrix} \quad \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

(9) $\quad \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0 \quad (1-\lambda)^2-4=0$

$(-1-\lambda)(3-\lambda)=0$

$\lambda=-1,\ 3$

$\lambda=-1$ $\quad \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad p+q=0 \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda=3$ $\quad \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad p-q=0 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

10) $\quad \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0 \quad (1-\lambda)^2=0$

$\lambda=1$ $\quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad q=0 \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

1, 2 $\quad A=\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \quad P=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad P^{-1}=\begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$

$B=P^{-1}AP=\begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 3 \\ -1 & 2 & -2 \\ -1 & 0 & 0 \end{pmatrix}$

$|B-\lambda E|=0 \quad \begin{vmatrix} 4-\lambda & 0 & 3 \\ -1 & 2-\lambda & -2 \\ -1 & 0 & -\lambda \end{vmatrix} = 0 \quad -\lambda^3+6\lambda^2-11\lambda+6=0$

$(\lambda-1)(\lambda-2)(\lambda-3)=0$

$\lambda=1,2,3$

$\lambda=1$ $\quad \begin{pmatrix} 3 & 0 & 3 \\ -1 & 1 & -2 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad \begin{matrix} p+r=0 \\ -p+q-2r=0 \end{matrix} \quad \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

$\lambda=2$ $\quad \begin{pmatrix} 2 & 0 & 3 \\ -1 & 0 & -2 \\ -1 & 0 & -2 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad \begin{matrix} 2p+3r=0 \\ p+2r=0 \end{matrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda=3$ $\quad \begin{pmatrix} 1 & 0 & 3 \\ -1 & -1 & -2 \\ -1 & 0 & -3 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad \begin{matrix} p+3r=0 \\ p+q+2r=0 \end{matrix} \quad \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$

13.3 (1) $\begin{vmatrix} x-2 & -1 & 1 \\ 1 & x-4 & 1 \\ 1 & -1 & x-2 \end{vmatrix} = (x-2)^2(x-4) - 2 + 2(x-2) - (x-4)$

$= (x-2)^2(x-4) + (x-2)$

$= (x-2)(x^2-6x+9) = (x-2)(x-3)^2$

(2) $\lambda = 2, 3$

$\lambda = 2$ $\begin{pmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $q = r$, $p = q$, $p - 2q + r = 0$ $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = 3$ $\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $p - q + r = 0$ $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

13.4 (1) 2次の直交行列を $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ とおくと ${}^tA A = E$ より

$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore$ $a^2+c^2=1$, $b^2+d^2=1$ $ab+cd=0$

$\therefore$ $a=\cos\theta$ とおくと $c=\sin\theta$ $d=\cos\theta$ $b=-\sin\theta$

$\therefore$ $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

(2) $\begin{vmatrix} \cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda \end{vmatrix} = 0$ $(\lambda-\cos\theta)^2 + \sin^2\theta = 0$

$\lambda^2 - 2\cos\theta\,\lambda + 1 = 0$

$\therefore$ $\lambda = \cos\theta \pm \sqrt{\cos^2\theta - 1} = \cos\theta \pm i\sin\theta$

$|\lambda| = 1$

13.5 は 12.12 と同じ. 13.6 は 12.7 と同じ

13.7 $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

(1) $\det A = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 8$

(2) $A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ -\frac{1}{4} & \frac{1}{8} & \frac{5}{8} \end{pmatrix}$

$= \frac{1}{8}\begin{pmatrix} 4 & -2 & -2 \\ -2 & 5 & 1 \\ -2 & 1 & 5 \end{pmatrix}$

3	1	1	1	0	0
1	2	0	0	1	0
1	0	2	0	0	1
1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0
0	$\frac{5}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	1	0
0	$-\frac{1}{3}$	$\frac{5}{3}$	$-\frac{1}{3}$	0	1
1	0	2	0	0	1
0	$\frac{5}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	1	0
0	0	$\frac{8}{5}$	$-\frac{2}{5}$	$\frac{1}{5}$	1
1	0	0	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$
0	$\frac{5}{3}$	0	$-\frac{5}{12}$	$\frac{25}{24}$	$\frac{5}{24}$
0	0	1	$-\frac{1}{4}$	$\frac{1}{8}$	$\frac{5}{8}$

(3) $\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$ $\quad (2-\lambda)^2(3-\lambda) - 2(2-\lambda) = 0$

$(2-\lambda)(\lambda^2 - 5\lambda + 4) = 0$

$(\lambda-2)(\lambda-1)(\lambda-4) = 0$

$\lambda = 1 \quad 2 \quad 4$

$\lambda = 1 \quad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} 2p+q+r=0 \\ p+q=0 \\ p+r=0 \end{array} \quad \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

$\lambda = 2 \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} p=0 \\ p+q+r=0 \end{array} \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$\lambda = 4 \quad \begin{pmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} -p+q+r=0 \\ p-2q=0 \\ p-2r=0 \end{array} \quad \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

13.8 $\quad A = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad |A - \lambda E| = 0 \quad \begin{vmatrix} -\lambda & 0 & 1 \\ -1 & -\lambda & 0 \\ 0 & -1 & -\lambda \end{vmatrix} = 0$

$-\lambda^3 + 1 = 0 \quad (\lambda-1)(\lambda^2+\lambda+1) = 0$

$\lambda = 1 \ , \ -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

$\lambda = 1 \quad \begin{pmatrix} -1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} -p+r=0 \\ p+q=0 \\ q+r=0 \end{array} \quad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

13.9 (1) $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{pmatrix}$ $\quad C = D = F = H = 0$ とおけば

$(A \ \ B)\begin{pmatrix} E \\ G \end{pmatrix} = (AE + BG)$

(2) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ とおくと

$\begin{vmatrix} I & O \\ A & I \end{vmatrix} = 1 \qquad \begin{pmatrix} I & O \\ A & I \end{pmatrix}^{-1} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$ とおく

$\begin{pmatrix} I & O \\ A & I \end{pmatrix}\begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} I & O \\ O & I \end{pmatrix} \quad \begin{pmatrix} E & F \\ AE+IG & AF+IH \end{pmatrix} = \begin{pmatrix} I & O \\ O & I \end{pmatrix}$

$\therefore \ E = I \quad F = O \quad A + G = O \quad H = I$

$\therefore \begin{pmatrix} I & O \\ A & I \end{pmatrix}^{-1} = \begin{pmatrix} I & O \\ -A & I \end{pmatrix}$

(3) $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} I & 0 \\ A & 2I+3A \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 5 & 6 \\ 1 & 0 & 3 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 1 & 2 & 5-\lambda & 6 \\ 1 & 0 & 3 & 2-\lambda \end{vmatrix} = 0. \quad (1-\lambda)^2\{(5-\lambda)(2-\lambda)-18\} = 0$$

$$(1-\lambda)^2(\lambda^2-7\lambda-8) = 0$$

$$(\lambda-1)^2(\lambda+1)(\lambda-8) = 0$$

$\lambda = 1,\ -1,\ 8$

$\lambda = 1$ $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 6 \\ 1 & 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $x+2y+4z+6w=0$, $x+3z+w=0$ $\begin{pmatrix} 2 \\ 5 \\ 0 \\ -2 \end{pmatrix}$ $\begin{pmatrix} 6 \\ 1 \\ -2 \\ 0 \end{pmatrix}$

$\lambda = -1$ $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 2 & 6 & 6 \\ 1 & 0 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $x=0$, $y=0$, $z+w=0$ $\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$

$\lambda = 8$ $\begin{pmatrix} -7 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 1 & 2 & -3 & 6 \\ 1 & 0 & 3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $x=y=0$, $z-2w=0$ $\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$

13.10 $f(x, Y) = |xE-AY| - |xE-YA|$

(1) (i) $|Y| \neq 0$ のとき

$$f(x,Y)|Y^{-1}| = \{|xE-AY| - |xE-YA|\}|Y^{-1}|$$
$$= |xY^{-1}-A| - |xY^{-1}-A| = 0$$

$\therefore f(x,Y) = 0 \quad \therefore |xE-AY| = |xE-YA|$

(ii) $|Y| = 0$ のとき $\mathrm{Rank}\,Y = m < n$ とする

Yの固有値は $\lambda_1, \lambda_2 \cdots \lambda_m$ $\lambda_i \neq 0$ $1 \leq i \leq m$

正則行列 P があって

$$P^{-1}YP = \left(\begin{array}{cc|c} \lambda_1 & 0 & 0 \\ 0 & \ddots\ \lambda_m & \\ \hline 0 & & 0 \end{array}\right) = \begin{pmatrix} H_m & 0 \\ 0 & 0 \end{pmatrix} \quad H_m = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & \\ 0 & & \ddots\ \lambda_m \end{pmatrix}$$

$$\begin{pmatrix} B & C \\ D & F \end{pmatrix}\begin{pmatrix} H_m & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} BH_m & 0 \\ DH_m & 0 \end{pmatrix}$$

$$\begin{pmatrix} H_m & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B & C \\ D & F \end{pmatrix} = \begin{pmatrix} H_mB & H_mC \\ 0 & 0 \end{pmatrix}$$

B : $m \times m$ 行列
D : $(n-m) \times m$ 行列
C : $m \times (n-m)$ 〃
F : $(n-m) \times (n-m)$

$|xE - AY| = |xE - P^{-1}APP^{-1}YP|$ $\quad P^{-1}APP^{-1}YP = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}$

$|xE - YA| = |xE - P^{-1}YPP^{-1}YP|$ $\quad P^{-1}YPP^{-1}AP = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$

$\therefore |xE - AY| - |xE - YA|$ $\quad P^{-1}AP = \begin{pmatrix} B & C \\ D & F \end{pmatrix}$ とおくと

$= x^{n-m}\{|xE - BH_m| - |xE - H_m B|\}$

$f(x, H_m) = |xE - BH_m| - |xE - H_m B\}$ $\quad |H_m| \neq 0$ により

$= 0$

$\therefore |xE - YA| = |xE - AY|$

(2) $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ $\quad A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \cdots & 0 \\ a_n & 0 & \cdots & 0 \end{pmatrix}$ $\quad Y = {}^tA = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & 0 & & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$

$AY = \begin{pmatrix} a_1^2 & a_1a_2 & a_1a_3 & \cdots & a_1a_n \\ a_2a_1 & a_2^2 & a_2a_3 & & a_2a_n \\ a_na_1 & a_na_2 & & & a_na_n \end{pmatrix}$ $\quad YA = \begin{pmatrix} \sum_{i=1}^{n} a_i^2 & 0 & \cdots & 0 \\ & & & \\ 0 & \cdots & & 0 \end{pmatrix}$

$\therefore |xE - \vec{a}\vec{a}^t| = |xE - \vec{a}^t\vec{a}| = x^{n-1}(x - \vec{a}^t\vec{a})$

(3) $B = \begin{pmatrix} n & -1 & -1 & \cdots & -1 \\ -1 & n & -1 & & -1 \\ -1 & -1 & n & & -1 \\ -1 & -1 & -1 & & n \end{pmatrix} = (n+1)E - \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$

$= (n+1)E - \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 1 & 0 & & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ $\quad \vec{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ とおくと

$= (n+1)E - \vec{a}\cdot\vec{a}^t$

$|xE - B| = |\{x - (n+1)\}E - (-\vec{a}\vec{a}^t)|$

$= (x - n - 1)^{n-1}(x - n - 1 + \vec{a}^t\vec{a})$

$= (x - n - 1)^{n-1}(x - 1)$

$\therefore x = 1 \quad x = n+1$ は $n-1$ 重根

nに無関係な固有値は 1

$x = 1$ $\quad \begin{pmatrix} 1-n & 1 & \cdots & 1 \\ 1 & 1-n & \cdots & 1 \\ 1 & 1 & & \\ 1 & 1 & \cdots & 1-n \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\quad \sum_{i=1}^{n} x_i = nx_j \quad 1 \leq j \leq n$

$\therefore x_j = 1$

固有ベクトルは ${}^t(1, 1, \cdots, 1)$

§14 固有空間

14.1 $A=\begin{pmatrix}1&a&b&c\\0&2&d&e\\0&0&2&0\\0&0&0&1\end{pmatrix}$ $|xE-A|=0$

$(x-1)^2(x-2)^2=0$

固有値 $x=1, 2$

$x=1$ 固有ベクトル $\begin{pmatrix}0&a&b&c\\0&1&d&e\\0&0&1&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ $ax_2+bx_3+cx_4=0$, $x_2+dx_3+ex_4=0$, $x_3=0$

$\therefore$ $ax_2+cx_4=0$

$x_2+ex_4=0$

$\therefore$ $ae-c=0$

$x=2$ 固有ベクトル $\begin{pmatrix}-1&a&b&c\\0&0&d&e\\0&0&0&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$ $-x_1+ax_2+bx_3+cx_4=0$, $dx_3+ex_4=0$, $-x_4=0$

$\therefore$ $ax_2+bx_3=x_1$

$dx_3=0$

$d\neq 0$ ならば $x_3=x_4=0$ $ax_2=x_1$ 固有ベクトル $\begin{pmatrix}a\\1\\0\\0\end{pmatrix}$ 1個

$\therefore$ $d=0$ ならば $\begin{pmatrix}a\\1\\0\\0\end{pmatrix}$ $\begin{pmatrix}a+b\\1\\1\\0\end{pmatrix}$

$\therefore$ $ae-c=0$ $d=0$ $b\neq 0$

14.2 $\begin{pmatrix}k&1&1\\1&k&1\\1&1&k\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

(1) $\begin{vmatrix}k&1&1\\1&k&1\\1&1&k\end{vmatrix}=0$ $k^3-3k+2=0$ $(k-1)^2(k+2)=0$

$k=1$ -2

(2) $k=1$ のとき

$x+y+z=0$ 原点を通り方向余弦$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$の平面

$k=-2$ のとき

$\begin{cases}-2x+y+z=0\\x-2y+z=0\\x+y-2z=0\end{cases}$ $x=y=z$

$\therefore$ 直線 $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$

14.3 $|xE-A|=0$ $\begin{vmatrix}x&0&-1\\-1&x&0\\0&-1&x\end{vmatrix}=0$ $x^3-1=0$ $(x-1)(x^2+x+1)=0$

(1) $\begin{pmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $x-z=0$, $-x+y=0$, $-y+z=0$ 固有ベクトル $\begin{pmatrix}1\\1\\1\end{pmatrix}$

$V = \{ (x, y, z) \mid x = y = z \}$

(2) $\vec{y} \in R^3$　$\vec{y} \perp V$　$\vec{y} \cdot (1, 1, 1) = 0$　$\vec{y} = (x, y, z)$ とおくと

$x + y + z = 0$

$$A\vec{y} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ x \\ y \end{pmatrix}$$

$$\cos\theta = \frac{zx + yx + yz}{x^2 + y^2 + z^2} = \frac{xz + yz + zx}{(x+y+z)^2 - 2(xy + yz + zx)} = -\frac{1}{2}$$

$\therefore \theta = \frac{2}{3}\pi$

14.4　$A = {}^tA$　$B = {}^tB$　$|C| \neq 0$

(1) $|xE - A| = 0$　$|xE - C^{-1}AC| = 0$　$|xE - B| = 0$

(2)

$$W = \left\{ X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \middle| AX = \lambda X \right\} \quad W' = \left\{ Y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \middle| AY = \lambda Y \right\}$$

Aの固有値を $\lambda_1, \lambda_2, \cdots \lambda_n$ とすると $|C| \neq 0$ で

$$C^{-1}AC = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$ となる C が存在する.

R^n で座標変換 C を行うと

R^n の基底 $\vec{e_i} = {}^t(0 \cdots \overset{i}{1} \cdots 0)$

$A\vec{e_i} = \lambda_i \vec{e_i}$

$\lambda = \lambda_1$　$\lambda' = \lambda_k$　$k \neq 1$ のとき

$W \ni X = x\,{}^t(1, 0 \cdots 0)$　$W' \ni Y = y\,{}^t(0 \cdots \overset{k}{1} \cdots 0)$

$X \cdot Y = 0$

$\therefore W \perp W'$

4.5　$|A - xE| = 0$　$\begin{vmatrix} 2-x & 1 & 1 \\ 1 & 2-x & 1 \\ 1 & 1 & 2-x \end{vmatrix} = 0$　$(2-x)^3 - 3(2-x) + 2 = 0$

$(2-x-1)^2(2-x+2) = 0$

$(1-x)^2(4-x) = 0$

$x = 1, 4$

$x = 1$　$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$　$p + q + r = 0$　$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$x = 4$　$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$　$-2p + q + r = 0$, $p - 2q + r = 0$, $p + q - 2r = 0$　$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

§15 Jordanの標準形

15.1 (1) $\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0 \quad (3-\lambda)^2 - 1 = 0 \quad \therefore \lambda = 2, 4$

$\lambda=2,\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \underline{\underline{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}} \quad \lambda=4,\ \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \underline{\underline{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}}$

(2) $|C-\lambda E|=0$ ならば $|PCP^{-1}-\lambda E|=0$

逆に $|P^{-1}CP-\lambda E|=0$ ならば $|C-\lambda E|=0$ ∴ 一致する.

(3) $P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

15.3 (1) $A = \begin{pmatrix} 3 & 2 & -2 \\ 2 & 3 & -2 \\ -2 & -2 & 3 \end{pmatrix} \quad |A-\lambda E| = 0 \quad \begin{vmatrix} 3-\lambda & 2 & -2 \\ 2 & 3-\lambda & -2 \\ -2 & -2 & 3-\lambda \end{vmatrix} = 0$

$(3-\lambda)^3 - 12(3-\lambda) + 16 = 0 \quad (3-\lambda-2)^2(3-\lambda+4) = 0$

$\lambda = 1, 7$

$\lambda = 1,\ \begin{pmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad p+q-r=0 \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = 7 \quad \begin{pmatrix} -4 & 2 & -2 \\ 2 & -4 & -2 \\ -2 & -2 & -4 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} -2p+q-r=0 \\ p-2q-r=0 \\ p+q+2r=0 \end{matrix} \quad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

(2) $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

15.4 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 4 & -1 \\ 3 & 0 \end{pmatrix}$

(1) $|A-\lambda E| = 0$

$(2-\lambda)^2 - 1 = 0$

$\therefore \lambda = 1, 3$

$\lambda = 1 \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad p+q=0 \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = 3 \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad p-q=0 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$|B-\lambda E| = 0 \quad -\lambda(4-\lambda)+3=0 \quad \lambda^2 - 4\lambda + 3 = 0$

$\lambda = 1, 3$

$\lambda = 1 \quad \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\lambda = 3 \quad \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(2) $|A-\lambda E| = |PAP^{-1}-\lambda E|$ 　同じ

(3) $P_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ 　$P_2 = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$ とおくと

$P_1^{-1} A P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ 　$P_2^{-1} B P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$\therefore P_1^{-1} A P_1 = P_2^{-1} B P_2$ 　　$B = P_2 P_1^{-1} A P_1 P_2^{-1} = (P_1 P_2^{-1})^{-1} A (P_1 P_2^{-1})$

$P_2^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 3 & -1 \end{pmatrix}$

$P = P_1 P_2^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 3 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$

15.5 (1) $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$ 　$|A-\lambda E| = 0$ 　$(3-\lambda)^2 - 1 = 0$

$\lambda^2 - 6\lambda + 8 = 0$ 　$\lambda = 2, 4$

(2) $\lambda = 2$ 　$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ 　$p - q = 0$ 　$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 4$ 　$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ 　$p + q = 0$ 　$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(3) $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ とおくと

$P^{-1} A P = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$

5.6 $\begin{pmatrix} 0 & -1 \\ 3 & 2 \end{pmatrix}$ 　$|A-\lambda E| = 0$ 　$-\lambda(2-\lambda) + 3 = 0$ 　$\lambda^2 - 2\lambda + 3 = 0$

$\lambda = 1 \pm \sqrt{2} i$

$\lambda = 1+\sqrt{2} i$ 　$\begin{pmatrix} -1-\sqrt{2} i & -1 \\ 3 & 1-\sqrt{2} i \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ 　$(1+\sqrt{2} i)p + q = 0$ 　$\begin{pmatrix} -1 \\ 1+\sqrt{2} i \end{pmatrix}$

$\lambda = 1-\sqrt{2} i$ 　$\begin{pmatrix} -1+\sqrt{2} i & -1 \\ 3 & 1+\sqrt{2} i \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ 　$(-1+\sqrt{2} i) - q = 0$ 　$\begin{pmatrix} 1 \\ -1+\sqrt{2} i \end{pmatrix}$

$P = \begin{pmatrix} -1 & 1 \\ 1+\sqrt{2} i & -1+\sqrt{2} i \end{pmatrix}$ とおくと

$P^{-1} A P = \begin{pmatrix} 1+\sqrt{2} i & 0 \\ 0 & 1-\sqrt{2} i \end{pmatrix}$

15.7 $A=\begin{pmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{pmatrix}$

(1) $\det A=\begin{vmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{vmatrix}=0$

(2) $|A-\lambda E|=0$ $\begin{vmatrix}1-\lambda&-1&0\\-1&2-\lambda&-1\\0&-1&1-\lambda\end{vmatrix}=0$ $(1-\lambda)^2(2-\lambda)-2(1-\lambda)=0$

$\lambda(1-\lambda)(-3+\lambda)=0$

$\lambda=0.1.3$

$\lambda=0$ $\begin{pmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{pmatrix}\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\begin{pmatrix}1\\1\\1\end{pmatrix}$

$\lambda=1$ $\begin{pmatrix}0&-1&0\\-1&1&-1\\0&-1&0\end{pmatrix}\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\begin{pmatrix}1\\0\\-1\end{pmatrix}$

$\lambda=3$ $\begin{pmatrix}-2&-1&0\\-1&-1&-1\\0&-1&-2\end{pmatrix}\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\begin{pmatrix}1\\-2\\1\end{pmatrix}$

$P=\begin{pmatrix}1&1&1\\1&0&-2\\1&-1&1\end{pmatrix}$

(3) $W=\begin{pmatrix}\frac{1}{\sqrt3}&\frac{1}{\sqrt2}&\frac{1}{\sqrt6}\\\frac{1}{\sqrt3}&0&-\frac{2}{\sqrt6}\\\frac{1}{\sqrt3}&-\frac{1}{\sqrt2}&\frac{1}{\sqrt6}\end{pmatrix}$ とおくと $AW=W\Lambda$ $W^tW=I$

$W^tW=\begin{pmatrix}\frac{1}{\sqrt3}&\frac{1}{\sqrt3}&\frac{1}{\sqrt3}\\\frac{1}{\sqrt2}&0&-\frac{1}{\sqrt2}\\\frac{1}{\sqrt6}&-\frac{2}{\sqrt6}&\frac{1}{\sqrt6}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt3}&\frac{1}{\sqrt2}&\frac{1}{\sqrt6}\\\frac{1}{\sqrt3}&0&\frac{-2}{\sqrt6}\\\frac{1}{\sqrt3}&-\frac{1}{\sqrt2}&\frac{1}{\sqrt6}\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=I$

$\Lambda=W^tAW=\begin{pmatrix}0&0&0\\0&1&0\\0&0&3\end{pmatrix}$

15.8 $A=\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$ 固有方程式 $(\lambda-\cos\theta)^2+\sin\theta=0$

$\lambda^2-2\lambda\cos\theta+1=0$ $\lambda=\cos\theta\pm i\sin\theta$

固有値は実数でないから実数の範囲では対角化できない

$\lambda=\cos\theta+i\sin\theta$ のとき $\begin{pmatrix}-i\sin\theta&-\sin\theta\\\sin\theta&-i\sin\theta\end{pmatrix}\begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ $\begin{pmatrix}1\\-i\end{pmatrix}$

$\lambda=\cos\theta-i\sin\theta$ $\begin{pmatrix}i\sin\theta&-\sin\theta\\\sin\theta&-i\sin\theta\end{pmatrix}\begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ $\begin{pmatrix}1\\i\end{pmatrix}$

$P=\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$ とおくとき　$P^{-1}AP=\begin{pmatrix} \cos\theta+i\sin\theta & 0 \\ 0 & \cos\theta-i\sin\theta \end{pmatrix}$

15.9　$A=\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 4 \end{pmatrix}$

(1) $\mathrm{Rank}\,A=\mathrm{Rank}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}=4$

(2) $A^{-1}=\frac{1}{4}\begin{pmatrix} 4 & -2 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$

(3) $\begin{vmatrix} 1-\lambda & 1 & 1 & 0 \\ 0 & 2-\lambda & 1 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 2 & 1 & 4-\lambda \end{vmatrix}=0$　　$(1-\lambda)^2(2-\lambda)(4-\lambda)=0$

$\lambda=1,\ 2,\ 4$

(4) $\lambda=1$　$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$　$y+z=0$, $2y+z+3w=0$　$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ -3 \\ 3 \\ 1 \end{pmatrix}$

$\lambda=2$　$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$　$z=0$, $-x+y=0$, $2y+2w=0$　$\begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda=4$　$\begin{pmatrix} -3 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$　$z=0$, $-3x+y=0$, $-2y+z=0$, $2y+z=0$　$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

(5) 対角化でき,

$P=\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}$ とおけば

$P^{-1}AP=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$

15.10 $A = \begin{pmatrix} 2 & 0 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$

(1) $\begin{vmatrix} 2-\lambda & 0 & 2 \\ 2 & 1-\lambda & 2 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$ $(1-\lambda)^2(2-\lambda) - 2(1-\lambda)$

$(1-\lambda)\lambda(\lambda-3) = 0$

$\lambda = 0, 1, 3$

$\lambda_1 = 0$, $\begin{pmatrix} 2 & 0 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $p + r = 0$ $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda_2 = 1$, $\begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $p + 2r = 0$, $2p + 2r = 0$, $p = 0$ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_3 = 3$ $\begin{pmatrix} -1 & 0 & 2 \\ 2 & -2 & 2 \\ 1 & 0 & -2 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $-p + 2r = 0$, $p - q + r = 0$ $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

(2) $T = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix}$

$T^{-1}AT = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(3) $\vec{x}(k) = \begin{pmatrix} x(k) \\ y(k) \\ z(k) \end{pmatrix} = A\vec{x}(k-1) = \begin{pmatrix} 2 & 0 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x(k-1) \\ y(k-1) \\ z(k-1) \end{pmatrix}$

$\vec{x}(1) = \begin{pmatrix} 2 & 0 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\vec{x}(2) = A\vec{x}(1) = A^2\vec{x}(0)$

$\therefore \vec{x}(k) = A^k \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$T^{-1}A^kT = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^k \end{pmatrix}$

$T = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix}$ $T^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 0 & -2 \\ -3 & 3 & -3 \\ 1 & 0 & 1 \end{pmatrix}$

$A^k = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^k \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & 0 & -2 \\ -3 & 3 & -3 \\ 1 & 0 & 1 \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 0 & 0 & 2\cdot 3^k \\ 0 & 1 & 3^{k+1} \\ 0 & 0 & 3^k \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ -3 & 3 & -3 \\ 1 & 0 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2\cdot 3^{k+1} & 0 & 2\cdot 3^{k+1} \\ -3+3^{k+1} & 3 & -3+3^{k+1} \\ 3^k & 0 & 3^k \end{pmatrix}$

$\vec{x}(k) = \begin{pmatrix} 2\cdot 3^k & 0 & 2\cdot 3^k \\ 1-3^k & 1 & -1+3^k \\ 3^{k-1} & 0 & 3^{k-1} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

15.11 $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$

a) $A^{-1} = \frac{1}{8}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

b) $\begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = 0 \quad (3-\lambda)^2 - 1 = 0 \quad (4-\lambda)(2-\lambda) = 0$

$\lambda = 2, \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad p - q = 0 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 4 \quad \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad p + q = 0 \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c) $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad P^t = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = P = P^{-1}$

$\therefore P^t A P = P^{-1} A P = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$

5.12 $A = \begin{pmatrix} a & a & 0 \\ a & 0 & a \\ 0 & a & a \end{pmatrix} \quad a \neq 0$

$$\left(\begin{array}{ccc|ccc} a & a & 0 & 1 & 0 & 0 \\ a & 0 & a & 0 & 1 & 0 \\ 0 & a & a & 0 & 0 & 1 \\ \hline a & a & 0 & 1 & 0 & 0 \\ 0 & a & -a & 1 & -1 & 0 \\ 0 & 0 & 2a & -1 & 1 & 1 \\ \hline a & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & a & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & a & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

(1) $A^{-1} = \frac{1}{2a}\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$

(2) $\begin{vmatrix} a-\lambda & a & 0 \\ a & -\lambda & a \\ 0 & a & a-\lambda \end{vmatrix} = 0 \quad -\lambda(a-\lambda)^2 - 2a^2(a-\lambda) = 0$

$(a-\lambda)(\lambda^2 - a\lambda - 2a^2) = 0$

$(a-\lambda)(\lambda - 2a)(\lambda + a) = 0$

$\lambda = a, -a, 2a$

$\lambda = a. \quad \begin{pmatrix} 0 & a & 0 \\ a & -a & a \\ 0 & a & 0 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} q = 0 \\ p + r = 0 \end{matrix} \quad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda = -a \quad \begin{pmatrix} 2a & a & 0 \\ a & a & a \\ 0 & a & 2a \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} 2p + q = 0 \\ p + q + r = 0 \\ q + 2r = 0 \end{matrix} \quad \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$\lambda = 2a \quad \begin{pmatrix} -a & a & 0 \\ a & -2a & a \\ 0 & a & -a \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} -p + q = 0 \\ p - 2q + r = 0 \\ q - r = 0 \end{matrix} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(3) $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \quad P^t = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} = P^{-1}$

${}^tP A P = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 2a \end{pmatrix}$

15.13 $A = \begin{pmatrix} a & -1 \\ a^2-a+1 & -3a \end{pmatrix}$ $\quad \begin{vmatrix} a-\lambda & -1 \\ a^2-a+1 & -3a-\lambda \end{vmatrix} = 0 \quad (\lambda-a)(\lambda+3a) + a^2-a+1 = 0$

$\lambda^2 + 2a\lambda - 2a^2 - a + 1 = 0$

$\frac{D}{4} = a^2 - (-2a^2 - a + 1) = 3a^2 + a - 1 > 0$

$3a^2 + a - 1 = 0 \quad a = \frac{-1 \pm \sqrt{13}}{6}$

$a > \frac{-1+\sqrt{13}}{6}$ or $a < \frac{-1-\sqrt{13}}{6}$

15.14 $A = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}$

(1) $\begin{vmatrix} -\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0 \quad \lambda(\lambda-2) - 3 = 0 \quad \lambda^2 - 2\lambda - 3 = 0$

$(\lambda-3)(\lambda+1) = 0$

$\lambda = -1, 3$

$\lambda = -1, \quad \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad p + q = 0 \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = 3 \quad \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad -3p + q = 0 \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

(2) $T = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$ とおく

$D = T^{-1}AT = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \qquad T^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$

(3) $D^n = T^{-1}A^nT = \begin{pmatrix} (-1)^n & 0 \\ 0 & 3^n \end{pmatrix}$

$A^n = T\begin{pmatrix} (-1)^n & 0 \\ 0 & 3^n \end{pmatrix}T^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 \\ 0 & 3^n \end{pmatrix}\frac{1}{4}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$

$= \frac{1}{4}\begin{pmatrix} (-1)^n & 3^n \\ (-1)^{n+1} & 3^{n+1} \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3(-1)^n + 3^n & (-1)^{n+1} + 3^n \\ (-1)^{n+1}3 + 3^{n+1} & (-1)^{n+2} + 3^{n+1} \end{pmatrix}$

§16 Jordan 標準形の応用

16.1 $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

(1) $\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$ $\quad (2-\lambda)^2(1-\lambda) - 2(2-\lambda) = 0$

$(2-\lambda)\lambda(\lambda-3) = 0$

(2) $\lambda = 0,\ 2,\ 3$

$\lambda = 0$ $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad \begin{matrix} 2p+r=0 \\ 2q+r=0 \\ p+q+r=0 \end{matrix}$ $\quad \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$\lambda = 2$ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad \begin{matrix} r=0 \\ p+q+r=0 \end{matrix}$ $\quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\lambda = 3$ $\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad \begin{matrix} -p+r=0 \\ -q+r=0 \\ p+q-2r=0 \end{matrix}$ $\quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$T = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$ とおく

$T^{-1}AT = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(3) $A^n = T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix} T^{-1}$

$= \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$

$= \begin{pmatrix} 2^{n-1}+3^{n-1} & -2^{n-1}+3^{n-1} & 3^{n-1} \\ -2^{n-1}+3^{n-1} & 2^{n-1}+3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{pmatrix}$

16.2 $A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$

(1) $|A - \lambda E| = 0$ $\begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = 0$ $\lambda(\lambda - 3) + 2 = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$\lambda = 1, 2$

$\lambda = 1$ $\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 2$ $\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $P^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

$B = P^{-1} A P = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

(2) $B^2 = P^{-1}APP^{-1}AP = P^{-1}A^2P$

$\therefore B^n = P^{-1}A^nP$

(3) $B^n = P^{-1}A^nP$ $\therefore \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix} = P^{-1}A^nP$

$A^n = P\begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix}P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2-2^n & -1+2^n \\ 2-2^{n+1} & -1+2^{n+1} \end{pmatrix}$

$\exp A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{2}{n!} - \sum_{n=0}^{\infty} \frac{2^n}{n!} & -\sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=0}^{\infty} \frac{2^n}{n!} \\ \sum \frac{2}{n!} - 2\sum \frac{2^n}{n!} & -\sum \frac{1}{n!} + 2\sum \frac{2^n}{n!} \end{pmatrix}$

$= \begin{pmatrix} 2e - e^2 & -e + e^2 \\ 2e - 2e^2 & -e + 2e^2 \end{pmatrix}$

16.3 $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & a \end{pmatrix}$

(1) Rank $A = 2$ $|A| = 0$ $\begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & a \end{vmatrix} = 0$ $2a - a - 1 = 0$

$a = 1$

$a = 1$

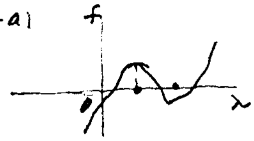
(2) $\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & a-\lambda \end{vmatrix} = 0$ $(1-\lambda)(2-\lambda)(a-\lambda) - (1-\lambda) - (a-\lambda) = 0$

$\lambda^3 - (a+3)\lambda^2 + 3a\lambda - a + 1 = 0$

$f(\lambda) = \lambda^3 - (a+3)\lambda^2 + 3a\lambda + (1-a)$

$f'(\lambda) = 3\lambda^2 - 2(a+3)\lambda + 3a$

$\frac{D}{4} = a^2 - 3a + 9 > 0$



$\therefore 1 - a < 0$

$a + 3 > 0$ $3a > 0$

$\therefore\ a>1$

(3) $x^2+2y^2+az^2-2xy-2yz=1$

$$(x\ \ y\ \ z)\begin{pmatrix}1&-1&0\\-1&2&-1\\0&-1&a\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=1$$

∴ Aの固有値を $\lambda_1, \lambda_2, \lambda_3$ としそれぞれの固有ベクトルを $\vec{p_1}, \vec{p_2}, \vec{p_3}$, $|\vec{p_1}|=|\vec{p_2}|=|\vec{p_3}|=1$ とするとき

$P=(\vec{p_1}\ \vec{p_2}\ \vec{p_3})$ とおき

$\begin{pmatrix}x\\y\\z\end{pmatrix}=P\begin{pmatrix}u\\v\\w\end{pmatrix}$ に変換すれば　${}^tPAP=\begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix}$

$(x, y, z)A\begin{pmatrix}x\\y\\z\end{pmatrix}=1$　$(u, v, w)\,{}^tPAP\begin{pmatrix}u\\v\\w\end{pmatrix}=1$

$\therefore\ \lambda_1u^2+\lambda_2v^2+\lambda_3w^2=1$　$\lambda_1, \lambda_2, \lambda_3$ がすべて正であれば

実だ円体∴(2)より $a>1$

16.4 (1) $A=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$ の固有値

$|A-\lambda E|=0$　$\begin{vmatrix}-\lambda&1&1\\1&-\lambda&1\\1&1&-\lambda\end{vmatrix}=-\lambda^3+2+3\lambda=0$

1	0	−3	−2	⌋−1
	−1	1	2	
1	−1	−2	0	
	−1	2		⌋
1	−2	0		

$\lambda^3-3\lambda-2=0$　$(\lambda+1)^2(\lambda-2)=0$

$\lambda=-1\ \ 2$

$\lambda=-1$　$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$　$p+q+r=0$　$\begin{pmatrix}1\\-2\\1\end{pmatrix}$ $\begin{pmatrix}1\\0\\-1\end{pmatrix}$

$\lambda=2$　$\begin{pmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{pmatrix}\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$　$-2p+q+r=0$, $p-2q+r=0$, $p+q-2r=0$　$\begin{pmatrix}1\\1\\1\end{pmatrix}$

(2) (1)より

$$T=\begin{pmatrix}\frac{1}{\sqrt6}&\frac{1}{\sqrt2}&\frac{1}{\sqrt3}\\-\frac{2}{\sqrt6}&0&\frac{1}{\sqrt3}\\\frac{1}{\sqrt6}&-\frac{1}{\sqrt2}&\frac{1}{\sqrt3}\end{pmatrix}$$

(3) $2xy+2zx+2xy+1=0$

$(x, y, z)\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-1$　$(x, y, z)A\begin{pmatrix}x\\y\\z\end{pmatrix}=-1$

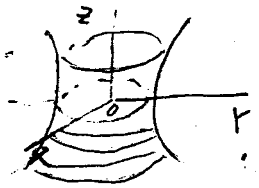
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = T \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$ とおくと

$$(X\ Y\ Z)\,{}^tTAT \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = -1 \qquad (X, Y, Z) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = -1$$

$\therefore\ -X^2 - Y^2 + 2Z^2 = -1$

$\therefore\ X^2 + Y^2 - 2Z^2 = 1$

$\therefore$ 回転双曲面



16.5 $g(x_1, x_2, x_3) = 2x_1^2 + x_2^2 + x_3^2 - 6x_2x_3$

(1) $$g(x_1, x_2, x_3) = (x_1\ x_2\ x_3) \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\therefore F = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & -3 & 1 \end{pmatrix}$$

(2) $|F - \lambda E| = 0 \qquad \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & -3 \\ 0 & -3 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)^2(2-\lambda) - 9(2-\lambda) = 0$

$(2-\lambda)(\lambda^2 - 2\lambda - 8) = 0$

$(2-\lambda)(\lambda-4)(\lambda+2) = 0$

$\lambda = -2,\ 2,\ 4$

$\lambda = -2,\quad \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad p = 0,\ q - r = 0 \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$\lambda = 2 \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -3 \\ 0 & -3 & -1 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} \quad q + 3r = 0,\ 3q + r = 0 \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = 4 \quad \begin{pmatrix} -2 & 0 & 0 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad p = 0,\ q + r = 0 \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

$$\vec{e}_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \vec{e}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{e}_3 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

(3) $$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad {}^tP = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = P^{-1}$$

$\vec{x} = P\vec{y}$

$$(x_1\ x_2\ x_3)\ F \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (y_1\ y_2\ y_3)\ {}^tP\,F\,P \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= (y_1\ y_2\ y_3) \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = -2y_1^2 + 2y_2^2 + 4y_3^2$$

16.6 $A = \begin{bmatrix} -1 & 3 & 3 \\ 2 & 1 & 0 \\ -2 & 1 & 2 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right)$$

(1) $P^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -2 & -3 \\ 0 & 1 & 1 \end{pmatrix}$

(2) $P^{-1}AP = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -2 & -3 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 & 3 \\ 2 & 1 & 0 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & -1 & -1 \\ 1 & -2 & -3 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & 4 \\ -1 & -1 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(3) $P^{-1}A^5P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{pmatrix}$

$$A^5 = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 1 & -2 & -3 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -2 & -3 \\ 0 & 32 & 32 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 33 & 33 \\ 2 & 61 & 60 \\ -2 & -29 & -28 \end{pmatrix}$$

§ 17 総合問題

7.1 $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$

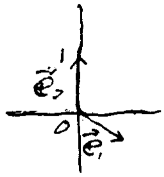
(1) $A^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$

(2) Rank $A = 2$

(3) $\begin{vmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = 0 \quad (1-\lambda)(2-\lambda) = 0$

$\lambda = 1, 2$

$\lambda = 1 \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad p + q = 0 \qquad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$



$\lambda = 2 \quad \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad p = 0 \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\vec{e}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(4) $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 1 \end{pmatrix}$ とおくと

$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \therefore P^{-1}A^{100}P = \begin{pmatrix} 1 & 0 \\ 0 & 2^{100} \end{pmatrix}$

$P^{-1} = \sqrt{2}\begin{pmatrix} 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ 1 & 1 \end{pmatrix}$

$\therefore A^{100} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^{100} \end{pmatrix}\begin{pmatrix} \sqrt{2} & 0 \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ -1+2^{100} & 2^{100} \end{pmatrix}$

7.2 $A = \begin{bmatrix} 1 & 3 & 2 & 6 & 3 \\ 4 & 5 & 5 & 4 & 2 \\ 0 & 1 & 3 & 2 & 0 \\ 3 & 1 & 0 & 5 & 3 \\ 3 & 0 & 0 & 4 & 5 \end{bmatrix}$

(1) $A^2 = \begin{bmatrix} 1 & 3 & 2 & 6 & 3 \\ 4 & 5 & 5 & 4 & 2 \\ 0 & 1 & 3 & 2 & 0 \\ 3 & 0 & 0 & 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 & 6 & 3 \\ 4 & 5 & 5 & 4 & 2 \\ 0 & 1 & 3 & 2 & 0 \\ 3 & 0 & 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 40 & 26 & 23 & 64 & 42 \\ 42 & 46 & 48 & 82 & 44 \\ 10 & 10 & 14 & 20 & 8 \\ 31 & 19 & 11 & 69 & 41 \\ 30 & 13 & 6 & 58 & 46 \end{bmatrix}$

(2) $|A| = \begin{vmatrix} 1 & 3 & 2 & 6 & 3 \\ 4 & 5 & 5 & 4 & 2 \\ 0 & 1 & 3 & 2 & 0 \\ 3 & 1 & 0 & 5 & 3 \\ 3 & 0 & 0 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -7 & 0 & 3 \\ 4 & 0 & -10 & -6 & 2 \\ 0 & 1 & 3 & 2 & 0 \\ 3 & 0 & -3 & 3 & 3 \\ 3 & 0 & 0 & 4 & 5 \end{vmatrix} = -\begin{vmatrix} 1 & -7 & 0 & 3 \\ 4 & -10 & -6 & 2 \\ 3 & -3 & 3 & 3 \\ 3 & 0 & 4 & 5 \end{vmatrix}$

$= -3\begin{vmatrix} 0 & -6 & -1 & 2 \\ 0 & -6 & -10 & -2 \\ 1 & 0 & -1 & 1 \\ 0 & 3 & 1 & 2 \end{vmatrix} = -3 \cdot \begin{vmatrix} -6 & -1 & 2 \\ -6 & -10 & -2 \\ 3 & 1 & 2 \end{vmatrix} = -3\begin{vmatrix} 0 & 1 & 6 \\ 0 & -8 & 2 \\ 3 & 1 & 2 \end{vmatrix} = -450$

(3) Rank $A = 5$

(4) $\det A \neq 0$: A^{-1}が存在する　正則行列である

17.3 $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \sqrt{3} \\ 1 & \sqrt{3} & -1 \end{pmatrix}$

(1) $\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & \sqrt{3} \\ 1 & \sqrt{3} & -1-\lambda \end{vmatrix} = 0$　　$-(1+\lambda)(1-\lambda)^2 - 4(1-\lambda) = 0$

$(1-\lambda)\{1-\lambda^2+4\} = 0$

$(1-\lambda)(5-\lambda^2) = 0$

$\lambda = 1,\ \sqrt{5},\ -\sqrt{5}$

$\lambda = 1$　$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & \sqrt{3} \\ 1 & \sqrt{3} & -2 \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix}$　$r = 0$, $p + \sqrt{3}q = 0$　$\begin{bmatrix} -\sqrt{3} \\ 1 \\ 0 \end{bmatrix}$

$\lambda = \sqrt{5}$　$\begin{bmatrix} 1-\sqrt{5} & 0 & 1 \\ 0 & 1-\sqrt{5} & \sqrt{3} \\ 1 & \sqrt{3} & -1-\sqrt{5} \end{bmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$　$(1-\sqrt{5})p + r = 0$, $(1-\sqrt{5})q + \sqrt{3}r = 0$, $p + \sqrt{3}q - (1+\sqrt{5})r = 0$　$\begin{bmatrix} 1 \\ \sqrt{3} \\ \sqrt{5}-1 \end{bmatrix}$

$\lambda = -\sqrt{5}$　$\begin{bmatrix} 1+\sqrt{5} & 0 & 1 \\ 0 & 1+\sqrt{5} & \sqrt{3} \\ 1 & \sqrt{3} & -1+\sqrt{5} \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$　$(1+\sqrt{5})p + r = 0$, $(1+\sqrt{5})q + \sqrt{3}r = 0$, $p + \sqrt{3}q + (-1+\sqrt{5})r = 0$　$\begin{bmatrix} 1 \\ \sqrt{3} \\ -1-\sqrt{5} \end{bmatrix}$

(2)

$A^{-1} = \frac{1}{5}\begin{bmatrix} 4 & -\sqrt{3} & 1 \\ -\sqrt{3} & 2 & \sqrt{3} \\ 1 & \sqrt{3} & -1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \sqrt{3} & 0 & 1 & 0 \\ 1 & \sqrt{3} & -1 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \sqrt{3} & 0 & 1 & 0 \\ 0 & \sqrt{3} & -2 & -1 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \sqrt{3} & 0 & 1 & 0 \\ 0 & 0 & -5 & -1 & -\sqrt{3} & 1 \\ \hline 1 & 0 & 0 & \frac{4}{5} & -\frac{\sqrt{3}}{5} & \frac{1}{5} \\ 0 & 1 & 0 & -\frac{\sqrt{3}}{5} & \frac{2}{5} & \frac{\sqrt{3}}{5} \\ 0 & 0 & 1 & \frac{1}{5} & \frac{\sqrt{3}}{5} & -\frac{1}{5} \end{array}\right]$$

$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1+2\sqrt{3} \\ 8 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & -\sqrt{3} & 1 \\ -\sqrt{3} & 2 & \sqrt{3} \\ 1 & \sqrt{3} & -1 \end{bmatrix}\begin{bmatrix} 1+2\sqrt{3} \\ 8 \\ 1 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} 1 \\ 10 \\ 10\sqrt{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ 2 \\ 2\sqrt{3} \end{bmatrix}$

7.4 (1) $A=\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ $A^2=\begin{pmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{pmatrix}$

$A^{-1}=\frac{1}{a^2-b^2}\begin{bmatrix} a & -b \\ -b & a \end{bmatrix}$ $(a^2 \neq b^2)$ $a^2=b^2$のとき A^{-1}は存在しない

(2) $A^n=\begin{bmatrix} c_n & d_n \\ d_n & c_n \end{bmatrix}$ $c_n=\frac{1}{2}\{(a+b)^n+(a-b)^n\}$

$d_n=\frac{1}{2}\{(a+b)^n-(a-b)^n\}$

[I] $n=1$のとき

$A^1=\begin{bmatrix} c_1 & d_1 \\ d_1 & c_1 \end{bmatrix}$ $c_1=\frac{1}{2}\{(a+b)+(a-b)\}=a$

$d_1=\frac{1}{2}\{(a+b)-(a-b)\}=b$

成り立つ

[II] $n=k$のとき成り立つと仮定すると

$A^k=\begin{pmatrix} c_k & d_k \\ d_k & c_k \end{pmatrix}$ $c_k=\frac{1}{2}\{(a+b)^k+(a-b)^k\}$

$d_k=\frac{1}{2}\{(a+b)^k-(a-b)^k\}$

$n=k+1$のとき

$A^{k+1}=A^k A=\begin{pmatrix} c_k & d_k \\ d_k & c_k \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix}$

$=\begin{bmatrix} c_k a+d_k b & c_k b+d_k a \\ c_k b+a d_k & c_k a+d_k b \end{bmatrix}$

$c_k a+d_k b=\frac{1}{2}\{(a+b)^k a+(a-b)^k a+(a+b)^k b-(a-b)^k b\}$

$=\frac{1}{2}\{(a+b)^{k+1}+(a-b)^{k+1}\}=c_{k+1}$

$c_k b+d_k a=\frac{1}{2}\{(a+b)^k b+(a-b)^k b+(a+b)^k a-(a-b)^k a\}$

$=\frac{1}{2}\{(a+b)^{k+1}-(a-b)^{k+1}\}=d_{k+1}$

$=\begin{pmatrix} c_{k+1} & d_{k+1} \\ d_{k+1} & c_{k+1} \end{pmatrix}$

∴ $n=k+1$のとき成り立つ

[I]，[II]よりすべての自然数について成り立つ

(3) $\begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = 0$ $(a-\lambda)^2 - b^2 = 0$

$(a+b-\lambda)(a-b-\lambda) = 0$

$\lambda = a+b,\ a-b$

$\lambda = a+b$ $\begin{pmatrix} -b & b \\ b & -b \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $p-q=0$ $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$\lambda = a-b$ $\begin{pmatrix} b & b \\ b & b \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $p+q=0$ $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

固有値 $a+b,\ a-b$

固有ベクトル $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix},\ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$