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第5章 積分法

§1. 不定積分

1.1 (1) $\int \log x\,dx = x\log x - x$

1.2 (1) $\int 2\frac{x^2}{x^2-1}dx = \int 2(1+\frac{1}{x^2-1})dx = \int(2+\frac{1}{x-1}-\frac{1}{x+1})dx = 2x + \log\left|\frac{x-1}{x+1}\right|$

(2) $\int \frac{\sqrt{x+1}}{x}dx = \int\frac{t\cdot 2t}{t^2-1}dt$ $\qquad \sqrt{x+1}=t \quad x=t^2-1 \quad dx=2t\,dt$

$= \int 2(1+\frac{1}{t^2-1})dt = \int(2+\frac{1}{t-1}-\frac{1}{t+1})dt = 2t + \log\left|\frac{t-1}{t+1}\right|$

$= 2\sqrt{x+1} + \log\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| = 2\sqrt{x+1} + 2\log|\sqrt{x+1}-1| - \log|x|$

(3) $\int\frac{x}{x^2-16}dx = \frac{1}{2}\log|x^2-16|$

(4) $\int x^2\sqrt{1-x^2}\,dx = \int \sin^2 t\cos^2 t\,dt$ $\qquad x=\sin t$ とおく $dx=\cos t\,dt$

$= \int\frac{1-\cos 2t}{2}\cdot\frac{1+\cos 2t}{2}dt = \frac{1}{4}\int(1-\cos^2 2t)dt = \frac{1}{4}\int\{1-\frac{1+\cos 4t}{2}\}dt$

$= \frac{1}{4}\{\frac{t}{2}-\frac{1}{8}\sin 4t\} = \frac{t}{8}-\frac{1}{32}\sin 4t = \frac{t}{8}-\frac{1}{16}\sin 2t\cos 2t$

$= \frac{t}{8}-\frac{1}{8}\sin t\cos t(1-2\sin^2 t) = \frac{t}{8}-\frac{1}{8}\sin t\cos t+\frac{1}{4}\sin^3 t\cos t$

$= \frac{1}{8}\sin^{-1}x - \frac{1}{8}x\sqrt{1-x^2}+\frac{1}{4}x^3\sqrt{1-x^2}$

(5) $\int xe^x dx = xe^x - e^x = (x-1)e^x$

(6) $\int\frac{1}{x}\log x\,dx = \frac{1}{2}(\log x)^2$

(7) $\int\frac{1}{\sqrt{4x^2+x+1}}dx$ $\qquad \sqrt{4x^2+x+1} = t-2x$ とおく

$x+1 = t^2-4xt \qquad x(4t+1)=t^2-1$

$= \int\frac{4t+1}{2t^2+t+2}\cdot\frac{2(2t^2+t+2)}{(4t+1)^2}dt$ $\qquad x=\frac{t^2-1}{4t+1}$

$\sqrt{4x^2+x+1} = t-\frac{2t^2-2}{4t+1} = \frac{2t^2+t+2}{4t+1}$

$= \int\frac{2}{4t+1}dt = \frac{1}{2}\log|4t+1|$ $\qquad dx = \frac{2t(4t+1)-4(t^2-1)}{(4t+1)^2}dt = \frac{4t^2+2t+4}{(4t+1)^2}dt$

$= \frac{1}{2}\log|4\sqrt{4x^2+x+1}+8x+1|$

(8) $\int\frac{1}{(1+e^x)^2}dx$ $\qquad e^x=t$ とおく $\quad x=\log t$

$dx=\frac{1}{t}dt$

$= \int\frac{1}{(1+t)^2 t}dt$

$= \int(\frac{1}{t}-\frac{1}{t+1}-\frac{1}{(t+1)^2})dt = \log t - \log(1+t)+\frac{1}{t+1}$

$= x-\log(1+e^x)+\frac{1}{1+e^x}$

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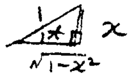
(9) $\int \frac{\sqrt{1-x^2}}{x^2}\,dx \qquad x = \sin t \qquad dx = \cos t\,dt$



$= \int \frac{\cos^2 t}{\sin^2 t}\,dt = \int \frac{1-\sin^2 t}{\sin^2 t}\,dt = \int \left(\frac{1}{\sin^2 t} - 1\right) dt = -\cot t - t$

$= -\frac{\sqrt{1-x^2}}{x} - \sin^{-1} x$

(10) $\int x\sqrt{1-x}\,dx \qquad \sqrt{1-x} = t \quad 1-x = t^2 \quad 1-t^2 = x$

$dx = -2t\,dt$

$= \int (1-t^2)\,t\,(-2t)\,dt$

$= -2\int (t^2 - t^4)\,dt = -2\left(\frac{t^3}{3} - \frac{t^5}{5}\right) = \left(-\frac{2}{3}t^2 + \frac{2}{5}t^4\right)t$

$= \left\{-\frac{2(1-x)}{3} + \frac{2}{5}(1-x)^2\right\}\sqrt{1-x} = \frac{1}{15}(6x^2 - 2x - 4)\sqrt{1-x}$

(11) $\int \log_a x\,dx = \int \frac{\log x}{\log a}\,dx = \frac{1}{\log a}\{x\log x - x\}$

(12) $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}$

(13) $\int \frac{e^x}{e^x+1}\,dx = \log(e^x+1)$

1.3 (1) $\int \tan^4 x \sec^2 x\,dx = \frac{1}{5}\tan^5 x$

(2) $\int x\tan^2 x\,dx = \int x(\sec^2 x - 1)\,dx = \int (x\sec^2 x - x)\,dx$

$= x\tan x - \int \tan x\,dx - \frac{x^2}{2} = x\tan x + \log|\cos x| - \frac{x^2}{2}$

(3) $\int e^{3x}\sin 2x\,dx = \frac{1}{3}e^{3x}\sin 2x - \frac{2}{3}\int e^{3x}\cos 2x\,dx$

$= \frac{1}{3}e^{3x}\sin 2x - \frac{2}{9}e^{3x}\cos 2x - \frac{4}{9}\int e^{3x}\sin 2x\,dx$

$\therefore \int e^{3x}\sin 2x\,dx = \frac{e^{3x}}{13}(3\sin 2x - 2\cos 2x)$

(4) $\int e^{-x}\sin x\,dx = -e^{-x}\sin x + \int e^{-x}\cos x\,dx$

$= -e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\sin x\,dx$

$\int e^{-x}\sin x\,dx = \frac{e^{-x}}{2}(-\sin x - \cos x)$

(5) $\int (\log x)^2\,dx = x(\log x)^2 - \int 2\log x\,dx$

$= x(\log x)^2 - 2x\log x + 2x$

(6) $\int e^{-x}\cos x\,dx = -e^{-x}\cos x - \int e^{-x}\sin x\,dx$

$= -e^{-x}\cos x + e^{-x}\sin x - \int e^{-x}\cos x\,dx$

$\therefore \int e^{-x}\cos x\,dx = \frac{e^{-x}}{2}(\sin x - \cos x)$

(7) $\int \frac{1}{1-\cos^2 x}\,dx = \int \frac{1}{\sin^2 x}\,dx = -\cot x$

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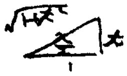
(8) $\int e^{ax}\sin bx\,dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx$

$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx\,dx$

$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

(9) $\int \frac{1}{2+\cos x}dx$ $\qquad \tan\frac{x}{2} = t$



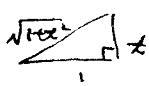
$x = 2\tan^{-1}t$

$= \int \frac{1}{2+\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}dt$ $\qquad dx = \frac{2}{1+t^2}dt$

$= \int \frac{2}{3+t^2}dt$ $\qquad \cos x = \frac{1-t^2}{1+t^2}$

$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right)$

(10) $\int \frac{1}{\sin x\cos x}dx = \int \frac{2}{\sin 2x}dx$ $\qquad \tan x = t$



$x = \tan^{-1}t$

$= \int \frac{2}{\frac{2t}{1+t^2}}\frac{1}{1+t^2}dt = \int\frac{1}{t}dt$ $\qquad dx = \frac{1}{1+t^2}dt$

$= \log t = \log|\tan x|$

(11) $\int \frac{1}{\cos x+\sin x}dx = \int \frac{1}{\frac{1-t^2+2t}{1+t^2}}\frac{2}{1+t^2}dt$ $\qquad \tan\frac{x}{2} = t$

$dx = \frac{2}{1+t^2}dt$

$= \int \frac{2}{1+2t-t^2}dt = -2\int\frac{1}{t^2-2t-1}dt$

$= -\int\frac{1}{\sqrt{2}}\left(\frac{1}{t-1-\sqrt{2}} - \frac{1}{t-1+\sqrt{2}}\right)dt = \frac{1}{\sqrt{2}}\log\left|\frac{t-1+\sqrt{2}}{t-1-\sqrt{2}}\right|$

$= \frac{1}{\sqrt{2}}\log\left|\frac{\tan\frac{x}{2}-1+\sqrt{2}}{\tan\frac{x}{2}-1-\sqrt{2}}\right|$

(12) $\int x\sin 3x\,dx = -\frac{x}{3}\cos 3x + \frac{1}{3}\int\cos 3x\,dx = -\frac{x}{3}\cos 3x + \frac{1}{9}\sin 3x$

(13) $\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx$

$= x^2\sin x + 2x\cos x - 2\int\cos x\,dx = x^2\sin x + 2x\cos x - 2\sin x$

(14) (2) に同じ

(15) $\int x\sin x\cos x\,dx = \int\frac{x}{2}\sin 2x\,dx = \frac{-x}{4}\cos 2x + \frac{1}{4}\int\cos 2x\,dx$

$= -\frac{x}{4}\cos 2x + \frac{1}{8}\sin 2x$

(16) $\int x\cos x\,dx = x\sin x - \int\sin x\,dx = x\sin x + \cos x$

1.4 (1) $\int \sin^{-1}x\,dx = x\sin^{-1}x - \int\frac{x}{\sqrt{1-x^2}}dx = x\sin^{-1}x + \sqrt{1-x^2}$

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(2) $\int \tan^{-1}x\,dx = x\tan^{-1}x - \int \frac{x}{1+x^2}dx = x\tan^{-1}x - \frac{1}{2}\log(1+x^2)$

(3) $y = \sec^{-1}x$ とおくと $x = \sec y$ $\quad \frac{dx}{dy} = \frac{\sin y}{\cos^2 y} = \sec y \tan y$

$= \sec y\sqrt{\sec^2 y - 1} = x\sqrt{x^2-1}$

$\therefore \frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$

$$\int \sec^{-1}x\,dx = x\sec^{-1}x - \int \frac{x}{x\sqrt{x^2-1}}dx = x\sec^{-1}x - \log|x+\sqrt{x^2-1}|$$

(4) (2) に同じ

1.5 (1) $\int \frac{\sqrt{1-x}}{x^2}dx$ $\qquad \sqrt{1-x} = t \quad 1-t^2 = x \quad dx = -2t\,dt$

$= \int \frac{t}{(1-t^2)^2}(-2t)dt = -\frac{t}{1-t^2} + \int \frac{1}{1-t^2}dt = -\frac{t}{1-t^2} + \int \frac{1}{2}\left(\frac{1}{1-t} + \frac{1}{1+t}\right)dt$

$= -\frac{t}{1-t^2} + \frac{1}{2}\log\left|\frac{1+t}{1-t}\right| = -\frac{t}{1-t^2} + \frac{1}{2}\log\frac{(1+t)^2}{1-t^2}$

$= -\frac{\sqrt{1-x}}{x} + \log|1+\sqrt{1-x}| - \frac{1}{2}\log|x|$

(2) $\int \frac{e^x(x^2-2x+1)}{(1+x^2)^2}dx = \int\left(\frac{e^x}{1+x^2} - \frac{2xe^x}{(1+x^2)^2}\right)dx$

$= \int \frac{e^x}{1+x^2}dx + \frac{e^x}{1+x^2} - \int \frac{e^x}{1+x^2}dx = \frac{e^x}{1+x^2}$

(3) $\int \sqrt{\frac{x+2}{x-3}}\,\frac{1}{x-2}dx$ $\qquad \sqrt{\frac{x+2}{x-3}} = t \quad x+2 = t^2(x-3)$

$(1-t^2)x = -(3t^2+2)$

$x = \frac{3t^2+2}{t^2-1} = 3 + \frac{5}{t^2-1}$

$dx = \frac{-10t}{(t^2-1)^2}dt$

$x-2 = \frac{3t^2+2}{t^2-1} - 2 = \frac{t^2+4}{t^2-1}$

$= \int t\cdot\frac{t^2-1}{t^2+4}\,\frac{-10t}{(t^2-1)^2}dt$

$= \int \frac{-10t^2}{(t^2+4)(t^2-1)}dt$

$= \int 2\left(\frac{1}{t^2-1} + \frac{4}{t^2+4}\right)dt$

$= \log\left|\frac{t+1}{t-1}\right| - 8\,\frac{1}{2}\tan^{-1}\frac{t}{2}$

$= \log\left|\frac{\sqrt{x+2}+\sqrt{x-3}}{\sqrt{x+2}-\sqrt{x-3}}\right| - 4\tan^{-1}\frac{1}{2}\sqrt{\frac{x+2}{x-3}}$

$= 2\log|\sqrt{x+2}+\sqrt{x-3}| - \log 5 - 4\tan^{-1}\frac{1}{2}\sqrt{\frac{x+2}{x-3}}$

(4) $\int \frac{1}{x+\sqrt{x^2+x+1}}dx$ $\qquad \sqrt{x^2+x+1} = t-x$ とおく

$x+1 = t^2-2xt \quad x(1+2t) = t^2-1$

$x = \frac{t^2-1}{2t+1} \quad dx = \frac{2t(2t+1)-2(t^2-1)}{(2t+1)^2}dt$

$= \int \frac{1}{t}\,\frac{2t^2+2t+2}{(2t+1)^2}dt$

$= \int\left(\frac{2}{t} - \frac{6t+6}{(2t+1)^2}\right)dt$

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(4) $$\int\left(\frac{2}{t}-\frac{6t+3+3}{(2t+1)^2}\right)dt=\int\left(\frac{2}{t}-\frac{3}{2t+1}-\frac{3}{(2t+1)^2}\right)dt$$

$$=2\log|t|-\frac{3}{2}\log|2t+1|+\frac{3}{2(2t+1)}$$

$$=2\log|x+\sqrt{x^2+x+1}|-\frac{3}{2}\log|2\sqrt{x^2+x+1}+2x+1|+\frac{3}{2}\frac{1}{\sqrt{x^2+x+1}+x}$$

$$=2\log|x+\sqrt{x^2+x+1}|-\frac{3}{2}\log|2\sqrt{x^2+x+1}+2x+1|+\frac{3(\sqrt{x^2+x+1}-x)}{2(x+1)}$$

1.6 (1) $$\int\frac{5x^2+3x+4}{x^3+x^2+x+1}dx=\int\frac{5x^2+3x+4}{(x^2+1)(x+1)}dx=\int\left(\frac{2x+1}{x^2+1}+\frac{3}{x+1}\right)dx$$

$$=\log|x^2+1|+\tan^{-1}x+3\log|x+1|$$

(2) $$\int\frac{5}{x^4-3x^2-4}dx=\int\left(\frac{1}{x^2-4}-\frac{1}{x^2+1}\right)dx=\int\frac{1}{4}\left(\frac{1}{x-2}-\frac{1}{x+2}\right)dx-\tan^{-1}x$$

$$=\frac{1}{4}\log\left|\frac{x-2}{x+2}\right|-\tan^{-1}x$$

(3) $$\int\frac{2x+1}{x(x+1)(x+2)}=\int\left(\frac{1}{2x}+\frac{1}{x+1}-\frac{3}{2(x+2)}\right)dx$$

$$=\frac{1}{2}\log|x|+\log|x+1|-\frac{3}{2}\log|x+2|$$

$$=\frac{1}{2}\log\frac{x(x+1)^2}{(x+2)^3}$$

(4) $$\int\frac{2x+1}{x(x-1)(x+2)}dx=\int\left(\frac{-1}{2x}+\frac{1}{x-1}-\frac{1}{2(x+2)}\right)dx$$

$$=\frac{1}{2}\log\left|\frac{(x-1)^2}{x(x+2)}\right|$$

(5) $$\int\frac{4x+19}{(x^2+4)(x+1)^2}dx=\int\left(\frac{-2x-1}{x^2+4}+\frac{2}{x+1}+\frac{3}{(x+1)^2}\right)dx$$

$$=-\log|x^2+4|-\frac{1}{2}\tan^{-1}x+2\log|x+1|-\frac{3}{x+1}$$

$$=\log\frac{(x+1)^2}{x^2+4}-\frac{1}{2}\tan^{-1}x-\frac{3}{x+1}$$

(6) $$\int\frac{4x+19}{(x^2+4)(x+1)}dx=\int\left(\frac{-3x+7}{x^2+4}+\frac{3}{x+1}\right)dx$$

$$=-\frac{3}{2}\log(x^2+4)+\frac{7}{2}\tan^{-1}\frac{x}{2}+3\log|x+1|$$

$$=\frac{3}{2}\log\frac{(x+1)^2}{x^2+4}+\frac{7}{2}\tan^{-1}\frac{x}{2}$$

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(7) $\int \frac{1}{(x-1)(x-2)}dx = \int\left(\frac{1}{x-2} - \frac{1}{x-1}\right)dx = \log\left|\frac{x-2}{x-1}\right|$

(8) $\int \frac{x^2}{x^4-1}dx = \int\frac{1}{2}\left(\frac{1}{x^2-1} + \frac{1}{x^2+1}\right)dx = \int\left\{\frac{1}{4}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) + \frac{1}{2}\,\frac{1}{x^2+1}\right\}dx$

$= \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\tan^{-1}x$

(9) $\int \frac{x^6}{x^4-1}dx = \int\left(x^2 + \frac{x^2}{x^4-1}\right)dx = \frac{x^3}{3} + \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\tan^{-1}x$

(10) $\int \frac{3x}{x^2-x-2}dx = \int\left(\frac{2}{x-2} + \frac{1}{x+1}\right)dx = \log(x-2)^2(x+1)$

(11) $\int \frac{x^2-x+4}{(x-1)(x-2)}dx = \int\left(x+3 + \frac{-4}{x-1} + \frac{10}{x-2}\right)dx$

$= \frac{x^2}{2} + 3x - 4\log|x-1| + 10\log|x-2|$

1.7 $I_n = \int(\log x)^n dx = x(\log x)^n - n\int(\log x)^{n-1}dx$

$\therefore I_n = x(\log x)^n - n I_{n-1}$

$I_1 = x\log x - x \qquad I_2 = x(\log x)^2 - 2I_1 = x(\log x)^2 - 2x\log x + 2x$

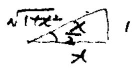
$I_3 = x(\log x)^3 - 3I_2 = x(\log x)^3 - 3x(\log x)^2 + 6x\log x - 6x$

$I_4 = x(\log x)^4 - 4I_3$

$= x(\log x)^4 - 4x(\log x)^3 + 12x(\log x)^2 - 24x\log x + 24x$

1.8 $\cot\frac{x}{2} = t \qquad 1 + \cot^2\frac{x}{2} = \frac{1}{\sin^2\frac{x}{2}} \qquad \therefore \sin^2\frac{x}{2} = \frac{1}{1+t^2} \quad \cos^2\frac{x}{2} = \frac{t^2}{1+t^2}$

$\cos x = \frac{t^2-1}{1+t^2} \qquad \frac{dt}{dx} = -\mathrm{cosec}^2\frac{x}{2}\cdot\frac{1}{2} = -\frac{1}{2\sin^2\frac{x}{2}}$



$-2\sin^2\frac{x}{2}\,dt = dx \qquad \frac{-2}{1+t^2}dt = dx$

$\int\frac{1}{1-\cos x}dx = \int\frac{1}{1-\frac{t^2-1}{1+t^2}}\,\frac{-2}{1+t^2}dt = \int -1\,dt = -t = -\cot\frac{x}{2}$

1.9 (1) $I_n = \int x(\log x)^n dx = \frac{x^2}{2}(\log x)^n - \frac{n}{2}\int x(\log x)^{n-1}dx$

$\therefore I_n = \frac{x^2}{2}(\log x)^n - \frac{n}{2}I_{n-1}$

(2) $I_n = \int\frac{1}{\cos^{2n}x}dx = \int\frac{1}{\cos^{2n-2}x}\sec^2x\,dx$

$= \tan x\,\frac{1}{\cos^{2n-2}x} - (2n-2)\int\tan x\,\frac{\sin x}{\cos^{2n-1}x}dx$

$= \tan x\,\frac{1}{\cos^{2n-2}x} - (2n-2)\int\frac{1-\cos^2x}{\cos^{2n}x}dx$

$= \frac{\sin x}{\cos^{2n-1}x} - (2n-2)I_n + (2n-2)I_{n-1}$

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§2 定積分

2.1 (1) $\int_0^2 \sqrt{3-x}\,dx = \cdot\int_3^0 t^{\frac{1}{2}}(-dt) = \int_0^3 t^{\frac{1}{2}}dt$ $\quad 3-x=t \quad dx=-dt$

$= \left[\frac{2}{3}t^{\frac{3}{2}}\right]_0^3 = 2\sqrt{3}$

(2) $\int_0^t \cos^2 x\,dx = \int_0^t \frac{1+\cos 2x}{2}dx = \left[\frac{x}{2}+\frac{1}{4}\sin 2x\right]_0^t = \frac{t}{2}+\frac{1}{4}\sin 2t$

(3) $\int_0^1 \frac{1}{x^2-x+1}dx = \int_0^1 \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}dx = \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{2x-1}{\sqrt{3}}\right]_0^1 = \frac{2}{\sqrt{3}}\left(\frac{\pi}{6}+\frac{\pi}{6}\right) = \frac{2\pi}{3\sqrt{3}}$

(4) $\int_1^2 \sin^2 \pi x\,dx = \int_1^2 \frac{1-\cos 2\pi x}{2}dx = \left[\frac{x}{2}-\frac{1}{4\pi}\sin 2\pi x\right]_1^2 = \frac{1}{2}$

(5) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}dx = \left[\tan^{-1}(\sin x)\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$

(6) $\int_0^1 \frac{1}{1+t}dt = \left[\log|1+t|\right]_0^1 = \log 2$

(7) $\int_0^2 \sqrt{4-x^2}\,dx = \int_0^{\frac{\pi}{2}} 4\cos^2 t\,dt = 4\cdot\frac{1}{2}\frac{\pi}{2} = \pi$ $\quad x=2\sin t$, $dx=2\cos t$

(8) $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{\pi}{4}$

(9) $\int_{-1}^1 x^2\sqrt{1+x^3}\,dx = \int_0^2 t^{\frac{1}{2}}\frac{1}{3}dt = \frac{2}{9}\left[t^{\frac{3}{2}}\right]_0^2$ $\quad 1+x^3=t$, $x^2dx=\frac{1}{3}dt$

$= \frac{4\sqrt{2}}{9}$

(10) $\int_0^2 \frac{\sqrt[4]{x}}{4(1+\sqrt{x})}dx$ $\quad \sqrt[4]{x}=t \quad x=t^4 \quad dx=4t^3dt$

$= \int_0^{\sqrt[4]{2}} \frac{t}{4(1+t^2)}4t^3dt = \int_0^{\sqrt[4]{2}}\frac{t^4}{1+t^2}dt = \int_0^{\sqrt[4]{2}}\left(t^2-1+\frac{1}{t^2+1}\right)dt$

$= \left[\frac{t^3}{3}-t+\tan^{-1}t\right]_0^{\sqrt[4]{2}} = \frac{1}{3}(\sqrt[4]{2})^3-\sqrt[4]{2}+\tan^{-1}\sqrt[4]{2}$

2.2 (1) $\int_0^1 x\log x\,dx = \left[\frac{x^2}{2}\log x-\frac{x^2}{4}\right]_0^1 = -\frac{1}{4}$

(2) $\int_0^\pi x\sin x\,dx = \left[-x\cos x\right]_0^\pi + \int_0^\pi \cos x\,dx = \pi + \left[\sin x\right]_0^\pi = \pi$

(3) $\int_1^3 \log x\,dx = \left[x\log x - x\right]_1^3 = 3\log 3 - 2$

(4) $\int_0^1 x\sin\frac{\pi x}{2}dx = \left[-\frac{2}{\pi}x\cos\frac{\pi}{2}x\right]_0^1 + \frac{2}{\pi}\int_0^1\cos\frac{\pi}{2}x\,dx$

$= \frac{4}{\pi^2}\left[\sin\frac{\pi}{2}x\right]_0^1 = \frac{4}{\pi^2}$

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(5) $\int_0^t x\sin x\,dx = [-x\cos x]_0^t + \int_0^t \cos x\,dx = -t\cos t + \sin t$

(6) $\int_0^2 x e^{\frac{x}{2}}dx = [2xe^{\frac{x}{2}}]_0^2 - 2\int_0^2 e^{\frac{x}{2}}dx = 4e - 4[e^{\frac{x}{2}}]_0^2$

$= 4e-(4e-4) = 4$

(7) $\int_1^2 x\log x\,dx = [\frac{x^2}{2}\log x - \frac{x^2}{4}]_1^2 = 2\log 2 - \frac{3}{4}$

2.3 (1) $\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos x}dx$ $\quad \tan\frac{x}{2} = t \quad \cos x = \frac{1-t^2}{1+t^2}$

$x = 2\tan^{-1}t$

$dx = \frac{2}{1+t^2}dt$

$= \int_0^1 \frac{1}{2-\frac{1-t^2}{1+t^2}}\,\frac{2}{1+t^2}dt$

$= \int_0^1 \frac{2}{3t^2+1}dt = \frac{2}{3}\int_0^1 \frac{1}{t^2+\frac{1}{3}}dt = \frac{2\sqrt{3}}{3}[\tan^{-1}\sqrt{3}t]_0^1$

$= \frac{2}{\sqrt{3}}\tan^{-1}\sqrt{3} = \frac{2\pi}{3\sqrt{3}}$

(2) $\int_0^{\frac{\pi}{4}} \frac{1}{3+\tan x}dx$ $\quad \tan x = t \quad x = \tan^{-1}t$

$dx = \frac{1}{1+t^2}dt$

$= \int_0^1 \frac{1}{(3+t)(1+t^2)}dt$

$= \int_0^1 \{\frac{1}{10(t+3)} - \frac{t}{10(t^2+1)} + \frac{3}{10(t^2+1)}\}dt$

$= [\frac{1}{10}\log|t+3| - \frac{1}{20}\log|t^2+1| + \frac{3}{10}\tan^{-1}t]_0^1$

$= \frac{1}{10}\log\frac{4}{3} - \frac{1}{20}\log 2 + \frac{3}{10}\frac{\pi}{4}$

$= \frac{3}{40}\pi - \frac{1}{20}\log 2\cdot\frac{3^2}{4^2} = \frac{3}{40}\pi - \frac{1}{20}\log\frac{9}{8}$

2.4 (1) $\int_0^\pi \frac{x\sin x}{a^2+\cos^2 x}dx = [-x\frac{1}{a}\tan^{-1}\frac{\cos x}{a}]_0^\pi + \frac{1}{a}\int_0^\pi \tan^{-1}\frac{\cos x}{a}dx$

$= \frac{\pi}{a}\tan^{-1}\frac{1}{a} + \frac{1}{a}\{\int_0^{\frac{\pi}{2}}\tan^{-1}\frac{\cos x}{a}dx + \int_{\frac{\pi}{2}}^\pi \tan^{-1}\frac{\cos x}{a}dx\}$

$= \frac{\pi}{a}\tan^{-1}\frac{1}{a} + \frac{1}{a}\{\int_0^{\frac{\pi}{2}}\tan^{-1}\frac{\cos x}{a}dx + \int_{\frac{\pi}{2}}^\pi \tan^{-1}\frac{\cos(\pi-t)}{a}dt\}$

$= \frac{\pi}{a}\tan^{-1}\frac{1}{a} + \frac{1}{a}\{\int_0^{\frac{\pi}{2}}\tan^{-1}\frac{\cos x}{a}dx - \int_0^{\frac{\pi}{2}}\tan^{-1}\frac{\cos t}{a}dt\}$

$= \frac{\pi}{a}\tan^{-1}\frac{1}{a}$

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2.5 $\int_1^{\sqrt{3}} \frac{1}{x^2(x^2+1)^2}dx = \int_1^{\sqrt{3}} \left(\frac{1}{x^2} - \frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}\right)dx$

$= \left[-\frac{1}{x} - \tan^{-1}x - \frac{1}{2}\frac{x}{x^2+1} - \frac{1}{2}\tan^{-1}x\right]_1^{\sqrt{3}}$

$= 1 - \frac{1}{\sqrt{3}} + \frac{\pi}{4} - \frac{\pi}{3} - \frac{\sqrt{3}}{8} + \frac{1}{4} - \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \frac{5}{4} - \frac{11\sqrt{3}}{24} - \frac{\pi}{8}$

2.6 (1) $\int_0^{x^2} f(t)\,dt$ 　　　　$\int f(t)\,dt = F(t)$ とおくと

$= [F(t)]_0^{x^2} = F(x^2) - F(0)$

$\therefore \frac{d}{dx}\int_0^{x^2} f(t)\,dt = 2x f(x^2)$

(2) $\frac{d}{dt}\int_0^t f(t,x)\,dx = f(t,x) + \int_0^t f_t(t,x)\,dx$

(3) $\frac{d}{dx}\int_{kx}^{x^2-kx} f(kt)\,dt = (2x-k) f(kx^2-k^2x) - k f(k^2x)$

2.7 $\{f(x)+t g(x)\}^2 = t^2\{g(x)\}^2 + 2f(x)g(x)t + \{f(x)\}^2 \geqq 0$

$\therefore t^2\int_a^b \{g(x)\}^2 dx + 2t\int_a^b f(x)g(x)\,dx + \int_a^b \{f(x)\}^2 dx \geqq 0$

$\therefore \left\{\int_a^b f(x)g(x)\,dx\right\}^2 \leqq \int_a^b \{f(x)\}^2 dx \int_a^b \{g(x)\}^2 dx$

"$=$" は $f(x) = g(x)$ のときのみ

2.8 $a > 0$ 　　$f(x) = f\left(\frac{a}{x}\right)$

$\int_1^a \frac{f(x)}{x}dx = \int_1^a \frac{f\left(\frac{a}{x}\right)}{x}dx$ 　　　$\sqrt{x} = t$ とおくと　$x = t^2$, $dx = 2t\,dt$

$= \int_1^{\sqrt{a}} \frac{f\left(\frac{a}{t^2}\right)}{t^2} 2t\,dt = 2\int_1^{\sqrt{a}} \frac{f\left(\frac{a}{t^2}\right)}{t}dt = 2\int_1^{\sqrt{a}} \frac{f(t^2)}{t}dt$

2.9 $\int_0^1 \frac{x^n}{x^4+4x^2+3}dx$

$n=0$ のとき　$\int_0^1 \frac{1}{x^4+4x^2+3}dx = \int_0^1 \left(\frac{1}{x^2+1} - \frac{1}{x^2+3}\right)\frac{1}{2}dt = \frac{1}{2}\left[\tan^{-1}t - \frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$

$= \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{\sqrt{3}}\frac{\pi}{6}\right) = \frac{3\sqrt{3}-2}{24\sqrt{3}}\pi$

$n=1$ $\displaystyle\int_0^1 \frac{x}{x^4+4x^2+3}\,dx = \int_0^1 \frac{1}{2}\left(\frac{x}{x^2+1} - \frac{x}{x^2+3}\right)dx = \frac{1}{4}\left[\log\frac{x^2+1}{x^2+3}\right]_0^1$

$\displaystyle = \frac{1}{4}\log\frac{2}{4}\cdot\frac{3}{1} = \frac{1}{4}\log\frac{3}{2}$

$n=2$ $\displaystyle\int_0^1 \frac{x^2}{x^4+4x^2+3}\,dx = \frac{1}{2}\int_0^1\left(\frac{3}{x^2+3} - \frac{1}{x^2+1}\right)dx$

$\displaystyle = \frac{1}{2}\left[\frac{3}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - \tan^{-1}x\right]_0^1 = \frac{\sqrt{3}}{2}\frac{\pi}{6} - \frac{1}{2}\frac{\pi}{4}$

$\displaystyle = \frac{2\sqrt{3}-3}{24}\pi$

$n=3$ $\displaystyle\int_0^1 \frac{x^3}{x^4+4x^2+3}\,dx = \frac{1}{2}\int_0^1\left(\frac{3x}{x^2+3} - \frac{x}{x^2+1}\right)dx$

$\displaystyle = \frac{1}{2}\left[\frac{3}{2}\log(x^2+3) - \frac{1}{2}\log(x^2+1)\right]_0^1$

$\displaystyle = \frac{3}{4}\log\frac{4}{3} - \frac{1}{4}\log 2 = \frac{1}{4}\log\frac{32}{27}$

2.10 $P(x) = \alpha x^2 + \beta x + \gamma$ とおく

$\displaystyle\int_a^b P(x)\,dx = \frac{\alpha}{3}(b^3-a^3) + \frac{\beta}{2}(b^2-a^2) + \gamma(b-a)$

$\displaystyle = \frac{b-a}{6}\{2\alpha b^2 + 2\alpha ba + 2\alpha a^2 + 3\beta b + 3\beta a + 6\gamma\}$

$\displaystyle = \frac{b-a}{6}\{P(b) + P(a) + \alpha(a+b)^2 + 2\beta(a+b) + 4\gamma\}$

$\displaystyle = \frac{b-a}{6}\left\{P(a) + P(b) + 4P\left(\frac{a+b}{2}\right)\right\}$

2.11 (1) $\displaystyle\int_0^x (x-t)f''(t)\,dt = \int_0^x x f''(t)\,dt - \int_0^x t f''(t)\,dt$

$\displaystyle = x f'(x) - \left\{\left[t f'(t)\right]_0^x - \int_0^x f'(t)\,dt\right\}$

$= x f'(x) - x f'(x) + f(x) = f(x)$

(2) $|f''(x)| < x \quad (x>0)$

$\displaystyle |f(x)| = \left|\int_0^x (x-t) f''(t)\,dt\right| \leq \int_0^x (x-t)|f''(t)|\,dt$

$\displaystyle \leq \int_0^x (x-t)t\,dt = \left[\frac{xt^2}{2} - \frac{t^3}{3}\right]_0^x = \frac{x^3}{6}$

$\displaystyle \therefore |f(x)| \leq \frac{x^3}{6}$

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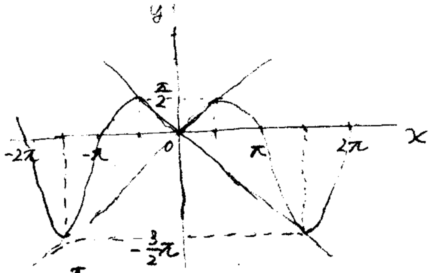
2.12 (1) $y = x \sin x$



(2) $\int_0^{\frac{\pi}{2}} x \sin x \, dx = \left[-x \cos x\right]_0^{\frac{\pi}{2}} + \left[\sin x\right]_0^{\frac{\pi}{2}} = 1$

2.13 $\int_0^{\frac{\pi}{2}} f(\sin \theta) d\theta$　　　$\sin \theta = t$ とおくと　$x = \sin^{-1} t$

$= \int_0^1 \frac{f(t)}{\sqrt{1-t^2}} dt$　　　$dx = \frac{1}{\sqrt{1-t^2}} dt$

2.14 $\int_0^{2\pi} \cos 3\theta (\cos \theta + \sin \theta) d\theta$

$= \int_0^{2\pi} \frac{1}{2} \{\cos 4\theta + \cos 2\theta + \sin 4\theta - \sin 2\theta\} d\theta$

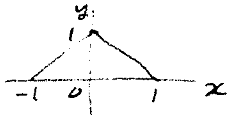
$= \frac{1}{2} \left[\frac{1}{4} \sin 4\theta + \frac{1}{2} \sin 2\theta - \frac{1}{4} \cos 4\theta + \frac{1}{2} \cos 2\theta\right]_0^{2\pi}$

$= 0$

P.48 §3 パラメータを含む定積分

3.1 $$f_n(x) = \begin{cases} 0 & |x| \geqq \frac{1}{n} \\ n(1-n|x|) & |x| \leqq \frac{1}{n} \end{cases}$$

(1) $$f_1(x) = \begin{cases} 1-|x| & |x|<1 \\ 0 & |x| \geqq 0 \end{cases}$$



(2) $$\int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\frac{1}{n}}^{\frac{1}{n}} n(1-n|x|)\,dx = 2n\int_0^{\frac{1}{n}} (1-nx)\,dx$$

$$= 2n\left[x - \frac{n}{2}x^2\right]_0^{\frac{1}{n}} = 2n\left(\frac{1}{n} - \frac{1}{2n}\right) = 1$$

(3) $$F_n(k) = \int_{-\infty}^{\infty} f_n(x) e^{-kx}\,dx = \int_{-\frac{1}{n}}^{\frac{1}{n}} n(1-n|x|) e^{-kx}\,dx$$

$$= \int_{-\frac{1}{n}}^{0} n(1+nx) e^{-kx}\,dx + \int_0^{\frac{1}{n}} n(1-nx) e^{-kx}\,dx$$

$$= \left[-\frac{n}{k} e^{-kx}\right]_{-\frac{1}{n}}^{0} - \left[\frac{n^2}{k} x e^{-kx}\right]_{-\frac{1}{n}}^{0} + \frac{n^2}{k}\int_{-\frac{1}{n}}^{0} e^{-kx}\,dx$$

$$+ \left[-\frac{n}{k} e^{-kx}\right]_0^{\frac{1}{n}} + \left[\frac{n^2}{k} x e^{-kx}\right]_0^{\frac{1}{n}} - \frac{n^2}{k}\int_0^{\frac{1}{n}} e^{-kx}\,dx$$

$$= -\frac{n}{k} + \frac{n}{k} e^{\frac{k}{n}} + \frac{n}{k} e^{\frac{k}{n}} - \left[\frac{n^2}{k^2} e^{-kx}\right]_{-\frac{1}{n}}^{0}$$

$$+ \frac{n}{k} - \frac{n}{k} e^{-\frac{k}{n}} + \frac{n}{k} e^{-\frac{k}{n}} + \left[\frac{n^2}{k^2} e^{-kx}\right]_0^{\frac{1}{n}}$$

$$= \frac{n^2}{k^2} e^{\frac{k}{n}} - \frac{n^2}{k^2} + \frac{n^2}{k^2} e^{-\frac{k}{n}} - \frac{n^2}{k^2}$$

$$= \frac{n^2}{k^2}\left(e^{\frac{k}{n}} + e^{-\frac{k}{n}} - 2\right)$$

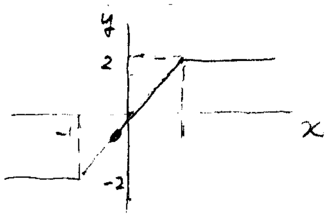
3.2 $$y(x) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos\theta \Big/ \sqrt{x^2+2x\sin\theta+1}\,d\theta$$

$x\sin\theta = t$ とおく

$x\cos\theta\,d\theta = dt$

$$= \int_{-x}^{x} \frac{dt}{\sqrt{x^2+1+2t}} = \left[\sqrt{x^2+1+2t}\right]_{-x}^{x}$$

$$= \sqrt{(x+1)^2} - \sqrt{(x-1)^2} = |x+1| - |x-1| = \begin{cases} -2 & x<-1 \\ 2x & -1<x<1 \\ 2 & x>1 \end{cases}$$



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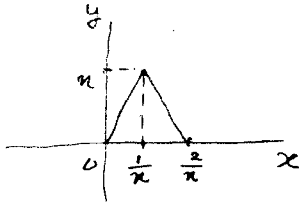
3.3 $f_n(x) = \begin{cases} n^2 x & 0 \leq x \leq \frac{1}{n} \\ 2n - n^2 x & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & x \leq 0,\ \frac{2}{n} \leq x \end{cases}$



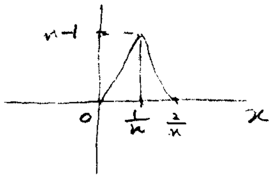
(1) $y = f_n(x)$ のグラフ

(2) $\lim\limits_{n\to\infty} f_n(x) = 0$

(3) $\int_0^1 f_n(x)\,dx = 1$

$\therefore \lim\limits_{n\to\infty} \int_0^1 f_n(x)\,dx = 1$. $\int_0^1 \lim f_n(x)\,dx = 0$

3.4 $f_n(x) = \begin{cases} n(n-1)x & 0 \leq x \leq \frac{1}{n} \\ 2(n-1) - n(n-1)x & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \frac{2}{n} \leq x \leq 1 \end{cases}$



(1) $\int_0^1 f_n(x)\,dx = 1 - \frac{1}{n}$

$\therefore \lim\limits_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim\limits_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1$

(2) $\int_0^1 \lim\limits_{n\to\infty} f_n(x)\,dx = 0$

3.5 (1) $\int_{-\pi}^{\pi} \cos nx \cos mx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \{\cos(m+n)x + \cos(n-m)x\}\,dx$

$= \frac{1}{2}\left[\frac{1}{m+n}\sin(m+n)x + \frac{1}{m-n}\sin(m-n)x\right]_{-\pi}^{\pi}$

$= \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$

(2) $\int_{-\pi}^{\pi} \sin nx \sin mx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \{\cos(n-m)x - \cos(n+m)x\}\,dx$

$= \frac{1}{2}\left[\frac{1}{n-m}\sin(n-m)x - \frac{1}{m+n}\sin(n+m)x\right]_{-\pi}^{\pi}$

$= \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$

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(8) $\int_{-\pi}^{\pi} \cos nx \sin mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \{ \sin(n+m)x - \sin(n-m)x \} dx$

$= \frac{1}{2} \left[\frac{-1}{n+m} \cos(m+n)x + \frac{1}{n-m} \cos(n-m)x \right]_{-\pi}^{\pi}$

$= \begin{cases} 0 & n \neq m \\ 0 & n = m \end{cases}$

3.6 $P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m} (x^2-1)^m$　　ルジャンドルの多項式

$\int_{-1}^{1} P_m(x) P_n(x) \, dx = \frac{1}{2^{m+n} m! n!} \int_{-1}^{1} \{(x^2-1)^m\}^{(m)} \{(x^2-1)^n\}^{(n)} dx$　　$\int (x^2-1)^{m(k)} dx = (x^2-1)^{m(k-1)}$

$\int_{-1}^{1} \{(x^2-1)^m\}^{(m)} \{(x^2-1)^n\}^{(n)} dx = \left[\{(x^2-1)^m\}^{(m-1)} \{(x^2-1)^n\}^{(n)} \right]_{-1}^{1} - \int_{-1}^{1} \{(x^2-1)^m\}^{(m-1)} \{(x^2-1)^n\}^{(n+1)} dx$

$= -\int_{-1}^{1} \{(x^2-1)^m\}^{(m-1)} \{(x^2-1)^n\}^{(n+1)} dx = (-1)^2 \int_{-1}^{1} \{(x^2-1)^m\}^{(m-2)} \{(x^2-1)^n\}^{(n+2)} dx$

$= (-1)^m \int_{-1}^{1} (x^2-1)^m \{(x^2-1)^n\}^{(n+m)} dx$

i) $n < m$ のとき $\{(x^2-1)^n\}^{(n+m)} = 0$

$\therefore \int_{-1}^{1} P_m(x) P_n(x) \, dx = 0$　　$n \neq m$

ii) $n = m$ のとき

$\int_{-1}^{1} \{P_n(x)\}^2 dx = \frac{1}{2^{2n} (n!)^2} (-1)^n \int_{-1}^{1} (x^2-1)^n \{(x^2-1)^n\}^{(2n)} dx$

$= \frac{(-1)^n}{2^{2n} (n!)^2} (2n)! \int_{-1}^{1} (x^2-1)^n dx$

$\int_{-1}^{1} (x^2-1)^n dx = \int_{-1}^{1} (x-1)^n (x+1)^n dx = \left[\frac{1}{n+1} (x+1)^{n+1} (x-1)^n \right]_{-1}^{1} - \frac{n}{n+1} \int_{-1}^{1} (x+1)^{n+1} (x-1)^{n-1} dx$

$= (-1)^n \frac{n!}{2n(2n-1)\cdots(n+1)} \int_{-1}^{1} (x+1)^{2n} dx$

$= (-1)^n \frac{n!}{2n(2n-1)\cdots(n+1)} \left[\frac{(x+1)^{2n+1}}{2n+1} \right]_{-1}^{1}$

$= (-1)^n \frac{(n!)^2 \, 2^{2n+1}}{(2n+1)!}$

$\therefore \int_{-1}^{1} \{P_n(x)\}^2 dx = \frac{2}{2n+1}$

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3.7 $\displaystyle\int_{-\pi}^{\pi} \sin(m+n)x\cos(m-n)x\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\{\sin 2mx + \sin 2nx\}\,dx$

$\displaystyle = \frac{1}{2}\left[\frac{-1}{2m}\cos 2mx - \frac{1}{2n}\cos 2nx\right]_{-\pi}^{\pi} = 0$

3.8 (1) $\displaystyle f(\alpha) = \int_0^1 |x^2 - \alpha^2|\,dx$

$\alpha^2 > 1$ のとき

$\displaystyle f(\alpha) = \int_0^1 (\alpha^2 - x^2)\,dx = \left[\alpha^2 x - \frac{x^3}{3}\right]_0^1 = \alpha^2 - \frac{1}{3}$

$\alpha^2 \leqq 1$ のとき

0 $|\alpha|$ 1

$\displaystyle f(\alpha) = \int_0^{|\alpha|}(\alpha^2 - x^2)\,dx + \int_{|\alpha|}^1 (x^2 - \alpha^2)\,dx$

$\displaystyle = \left[\alpha^2 x - \frac{x^3}{3}\right]_0^{|\alpha|} + \left[\frac{x^3}{3} - \alpha^2 x\right]_{|\alpha|}^1$

$\displaystyle = \alpha^2|\alpha| - \frac{\alpha^2}{3}|\alpha| + \frac{1}{3} - \alpha^2 - \frac{\alpha^2}{3}|\alpha| + \alpha^2|\alpha|$

$\displaystyle = \frac{4}{3}\alpha^2|\alpha| - \alpha^2 + \frac{1}{3}$

$|\alpha| > 1$ のとき $\displaystyle f(\alpha) = \alpha^2 - \frac{1}{3}$

$|\alpha| < 1$ のとき $\displaystyle f(\alpha) = \frac{4}{3}\alpha^2|\alpha| - \alpha^2 + \frac{1}{3}$

(2) $\displaystyle f(\alpha) = \frac{1}{4}$

$|\alpha| > 1$ とすれば $\displaystyle \alpha^2 - \frac{1}{3} = \frac{1}{4}$ $\displaystyle \alpha^2 = \frac{7}{12}$ 解なし

$|\alpha| < 1$ とすれば $\displaystyle \frac{4}{3}\alpha^3 - \alpha^2 + \frac{1}{3} = \frac{1}{4}$ $(\alpha > 0)$

$\displaystyle \frac{4}{3}\alpha^3 - \alpha^2 + \frac{1}{12} = 0$ $16\alpha^3 - 12\alpha^2 + 1 = 0$

$$\begin{array}{rrrr|l} 16 & -12 & 0 & 1 & \frac{1}{2} \\ & 8 & -2 & -1 & \\ \hline 16 & -4 & -2 & 0 & \\ & 8 & 2 & & \\ \hline 16 & 4 & 0 & & \end{array}$$

$\displaystyle (\alpha - \frac{1}{2})^2(16\alpha + 4) = 0$

$\displaystyle \alpha = \frac{1}{2},\ -\frac{1}{4}$

$\alpha < 0$ のとき $16\alpha^3 + 12\alpha^2 - 1 = 0$

$\displaystyle (\alpha + \frac{1}{2})^2(16\alpha - 4) = 0$

$\displaystyle \therefore \alpha = -\frac{1}{2},\ \frac{1}{4}$

$\displaystyle \therefore \alpha = \pm\frac{1}{2}$

P. 48　§4 漸化式による定積分

4.1 (1) $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$

$= \int_0^{\frac{\pi}{2}} (\sin^{n-2} x - \sin^{n-2} x \cos x \cos x)\, dx$

$= I_{n-2} - \frac{1}{n-1}\left[\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}} - \frac{1}{n-1}\int_0^{\frac{\pi}{2}} \sin^n x\, dx$

$\therefore \frac{n}{n-1} I_n = I_{n-2}$　$\therefore I_n = \frac{n-1}{n} I_{n-2}$　$I_0 = \frac{\pi}{2}$　$I_1 = 1$

$\therefore$ $n = 2m$ のとき　$I_{2m} = \frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdots\frac{1}{2} I_0$

$= \frac{2m-1}{2m}\cdot\frac{2m-1}{2m-2}\cdots\frac{1}{2}\frac{\pi}{2}$

$n = 2m-1$ のとき　$I_{2m-1} = \frac{2m-2}{2m-1}\,\frac{2m-4}{2m-3}\cdots\frac{2}{3}$

4.2 (1) $\int_0^{\frac{\pi}{2}} \sin^k x\, dx = \begin{cases} \frac{k-1}{k}\,\frac{k-3}{k-2}\cdots\frac{1}{2}\,\frac{\pi}{2} & k = 2m \text{ のとき} \\ \frac{k-1}{k}\,\frac{k-3}{k-2}\cdots\frac{2}{3} & k = 2m-1 \text{ のとき} \end{cases}$

(2) $I_{2n} = \frac{(2n-1)(2n-3)\cdots 1}{2n(2n-2)\cdots 2}\cdot\frac{\pi}{2}$

$\frac{2\cdot 4\cdots(2n-2)2n}{1\cdot 3\cdots(2n-1)} = \frac{\pi}{2 I_{2n}}$

$I_{2n+1} = \frac{2n(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 3}$

$\therefore \left\{\frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n-1)}\right\}^2 \frac{1}{2n+1} = \frac{\pi I_{2n+1}}{2 I_{2n}}$

$\therefore \frac{1}{n}\left\{\frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n-1)}\right\}^2 = \frac{\pi}{2}\cdot\frac{2n+1}{n}\cdot\frac{I_{2n+1}}{I_{2n}}$

$I_n - I_{n+1} = \int_0^{\frac{\pi}{2}} \sin^n x(1-\sin x)\, dx > 0$

$\therefore I_n > I_{n+1}$，$I_n > 0$　数列$\{I_n\}$は単調減少

$\therefore \lim_{n\to\infty}\frac{I_{n+1}}{I_n} = 1$

$\therefore \lim_{n\to\infty}\frac{1}{n}\left\{\frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n-1)}\right\}^2 = \lim_{n\to\infty}\frac{\pi}{2}\left(\frac{2n+1}{n}\right)\frac{I_{2n+1}}{I_{2n}} = \pi$

4.3 $I_{m\,n} = \int_0^1 x^m(1-x)^n dx = \left[\frac{1}{m+1}x^{m+1}(1-x)^n\right]_0^1 + \frac{n}{m+1}\int_0^1 x^{m+1}(1-x)^{n-1}dx$

$= \frac{n}{m+1} I_{m+1\,n-1} = \frac{n(n-1)}{(m+2)(m+1)} I_{m+2\,n-2} \cdots$

$= \frac{n!}{(m+1)(m+2)\cdots(m+n)} I_{m+n,\,0} = \frac{n!\,m!}{(m+n+1)!}$

$I_{m\,n-1} = \frac{(n-1)!\,m!}{(m+n)!}$　　$\therefore I_{m,n} = \frac{n}{m+n+1} I_{m,n-1}$

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4.4 $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$

(1) $\sin x = \cos\left(\frac{\pi}{2} - x\right)$

$\therefore I_n = \int_0^{\frac{\pi}{2}} \cos^n\left(\frac{\pi}{2} - x\right) dx = -\int_{\frac{\pi}{2}}^{0} \cos^n t \, dt$ 　　$\frac{\pi}{2} - x = t$ とおく

$= \int_0^{\frac{\pi}{2}} \cos^n t \, dt = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$

(2) $I_2 = \int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2x) \, dx = \frac{1}{2} \left[x - \frac{1}{2} \sin 2x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$

(3) $I_{n+2} = \int_0^{\frac{\pi}{2}} \sin^{n+2} x \, dx = \int_0^{\frac{\pi}{2}} (\sin^n x - \sin^n x \cos x \cos x) \, dx$

$= I_n - \left[\frac{1}{n+1} \sin^{n+1} x \cos x\right]_0^{\frac{\pi}{2}} - \frac{1}{n+1} \int_0^{\frac{\pi}{2}} \sin^{n+2} x \, dx$

$\therefore \left(1 + \frac{1}{n+1}\right) I_{n+2} = I_n$

$\therefore I_{n+2} = \frac{n+1}{n+2} I_n$

(4) $$I_n = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \cdot \frac{1}{2} \frac{\pi}{2} & n = 2m \\ \frac{n-1}{n} \frac{n-3}{n-2} \cdots \cdot \frac{2}{3} & n = 2m-1 \end{cases}$$

4.5 $I_n = \int_1^e (\log x)^n dx$

(1) $I_1 = \int_1^e \log x \, dx = [x \log x - x]_1^e = e - (e-1) = 1$

(2) $I_n = \int_1^e (\log x)^n dx = [x (\log x)^n]_1^e - \int n x \frac{1}{x} (\log x)^{n-1} dx$

$= e - n I_{n-1}$

(3) $I_4 = e - 4 I_3 = e - 4(e - 3 I_2) = -3e + 12(e - 2 I_1)$

$= 9e - 24$

4.6 $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, (\sec^2 x - 1) \, dx$

$= \left[\frac{1}{n-1} \tan^{n-1} x\right]_0^{\frac{\pi}{4}} - I_{n-2} = \frac{1}{n-1} - I_{n-2}$

$I_1 = \int_0^{\frac{\pi}{4}} \tan x \, dx = [-\log(\cos x)]_0^{\frac{\pi}{4}} = -\log \frac{1}{\sqrt{2}} = \frac{1}{2} \log 2$

$I_0 = \int_0^{\frac{\pi}{4}} 1 \, dx = \frac{\pi}{4}$

$I_2 = \int_0^{\frac{\pi}{4}} \tan^2 x \, dx = \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) \, dx = 1 - \frac{\pi}{4}$

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$$I_n = \begin{cases} \dfrac{1}{n-1} - \dfrac{1}{n-2} + \cdots + (-1)^{\frac{n}{2}-1}\cdot 1 + (-1)^{\frac{n}{2}}\dfrac{\pi}{4} & (n：偶数) \\ \dfrac{1}{n-1} - \dfrac{1}{n-2} + \cdots + (-1)^{\frac{n-1}{2}-1}\dfrac{1}{2} + (-1)^{\frac{n-1}{2}}\dfrac{1}{2}\log 2 & (n：奇数) \end{cases}$$

4.7 $$I_n = \int_0^1 (\log x)^n dx = \left[x(\log x)^n\right]_0^1 - n\int_0^1 (\log x)^{n-1} dx$$

$$\lim_{x\to 0} x\log x = \lim_{x\to 0}\frac{\log x}{\frac{1}{x}} = \lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$$

$$\lim_{x\to 0} x(\log x)^n = \lim_{x\to 0}\frac{(\log x)^n}{\frac{1}{x}} = \lim_{x\to 0}\frac{n(\log x)^{n-1}}{-\frac{1}{x}} = 0$$

$$\therefore I_n = -n I_{n-1}$$

$$I_1 = \int_0^1 (\log x)\,dx = \left[x\log x - x\right]_0^1 = -1$$

$$\therefore I_n = (-1)^{n-1} n(n-1)\cdots 2\, I_1 = (-1)^n n!$$

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§5 広義積分

5.1 (1) $\int_0^\infty \frac{1}{x^a}dx \quad (a>0)$

$= \lim_{\substack{\varepsilon\to 0\\ M\to\infty}} \left[\frac{1}{1-a}x^{1-a}\right]_\varepsilon^M = \lim_{\substack{\varepsilon\to 0\\ M\to\infty}} \frac{1}{1-a}\{M^{1-a}-\varepsilon^{1-a}\} = \begin{cases}\infty & a>1\\ \infty & a<1\end{cases}$

$a=1$ のとき $\int_0^\infty \frac{1}{x}dx = \lim_{\substack{\varepsilon\to 0\\ M\to\infty}}\left[\log|x|\right]_\varepsilon^M = \lim_{\substack{\varepsilon\to 0\\ M\to\infty}} \frac{M}{\varepsilon} = \infty$

$\therefore \int_0^\infty \frac{1}{x^a}dx$ の関数なし

(2) $\int_{-\infty}^\infty \frac{dx}{e^x+4e^{-x}+5} = \int_{-\infty}^\infty \frac{e^x}{e^{2x}+5e^x+4}dx \qquad e^x=t,\ x=\log t,\ dx=\frac{1}{t}dt$

$= \int_0^\infty \frac{1}{t^2+5t+4}dt = \int_0^\infty \frac{1}{3}\left(\frac{1}{t+1}-\frac{1}{t+4}\right)dt$

$= \frac{1}{3}\lim_{M\to\infty}\left[\log\left|\frac{t+1}{t+4}\right|\right]_0^M = \frac{-1}{3}\log\frac{1}{4} = \frac{2}{3}\log 2$

(3) $\int_0^\infty x^2e^{-x}dx = \left[-x^2e^{-x}\right]_0^\infty + 2\int_0^\infty xe^{-x}dx = \left[-2xe^{-x}\right]_0^\infty + 2\int_0^\infty e^{-x}dx$

$= \left[-2e^{-x}\right]_0^\infty = 2$

(4) $\int_0^\infty \frac{1}{\sqrt[3]{e^x-1}}dx \qquad \sqrt[3]{e^x-1}=t$ とおく $e^x=t^3+1$

$x=\log(t^3+1) \quad dx=\frac{3t^2}{t^3+1}dt$

$= \int_0^\infty \frac{1}{t}\frac{3t^2}{t^3+1}dt$

$= \int_0^\infty \frac{3t}{t^3+1}dt = \int_0^\infty\left(\frac{-1}{t+1}+\frac{t+1}{t^2-t+1}\right)dt$

$= \int_0^\infty\left\{\frac{-1}{t+1}+\frac{t-\frac{1}{2}}{t^2-t+1}+\frac{\frac{3}{2}}{(t-\frac{1}{2})^2+\frac{3}{4}}\right\}dt$

$= \left[-\log|t+1|+\frac{1}{2}\log|t^2-t+1|+\sqrt{3}\tan^{-1}\frac{2t-1}{\sqrt{3}}\right]_0^\infty$

$= \left[\frac{1}{2}\log\frac{t^2-t+1}{(t+1)^2}+\sqrt{3}\tan^{-1}\frac{2t-1}{\sqrt{3}}\right]_0^\infty$

$= \sqrt{3}\left(\frac{\pi}{2}+\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3}\pi$

(5) $\int_1^\infty \frac{1}{x(1+x^2)}dx = \int_1^\infty\left(\frac{1}{x}-\frac{x}{1+x^2}\right)dx = \left[\log|x|-\frac{1}{2}\log(1+x^2)\right]_1^\infty$

$= \frac{1}{2}\left[\log\frac{x^2}{1+x^2}\right]_1^\infty = -\frac{1}{2}\log\frac{1}{2} = \frac{1}{2}\log 2$

(6) $\int_0^\infty \frac{1}{e^x+e^{-x}}dx = \int_0^\infty \frac{e^x}{e^{2x}+1}dx = \left[\tan^{-1}e^x\right]_0^\infty = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4}$

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(7) $\int_0^\infty x^n e^{-x} dx = [-x^n e^{-x}]_0^\infty + n\int_0^\infty x^{n-1} e^{-x} dx = n\int_0^\infty x^{n-1} e^{-x} dx$

$= [-nx^{n-1} e^{-x}]_0^\infty + n(n-1)\int_0^\infty x^{n-2} e^{-x} dx$

$= n! \int_0^\infty e^{-x} dx = n!\ [-e^{-x}]_0^\infty = n!$

(8) $\int_0^\infty \frac{1}{x^3+1} dx = \int_0^\infty \{\frac{1}{3(x+1)} + \frac{-x+2}{3(x^2-x+1)}\} dx$

$= \frac{1}{3}\int_0^\infty \{\frac{1}{x+1} + \frac{-x+\frac{1}{2}}{x^2-x+1} + \frac{\frac{3}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}\} dx$

$= \frac{1}{3}\left[\log|x+1| - \frac{1}{2}\log|x^2-x+1| + \sqrt{3}\tan^{-1}\frac{2x-1}{\sqrt{3}}\right]_0^\infty$

$= \frac{1}{3}\{\left[\frac{1}{2}\log\frac{(x+1)^2}{x^2-x+1} + \sqrt{3}\tan^{-1}\frac{2x-1}{\sqrt{3}}\right]_0^\infty$

$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} + \frac{\pi}{6}\right) = \frac{\pi}{2\sqrt{3}}$

(9) $\int_0^\infty x e^{-ax} dx \qquad a > 0$

$= [-\frac{x}{a} e^{-ax}]_0^\infty + \frac{1}{a}\int_0^\infty e^{-ax} dx = [-\frac{1}{a^2} e^{-ax}]_0^\infty = \frac{1}{a^2}$

(10) $\int_0^\infty x e^{-x} dx = [-x e^{-x}]_0^\infty + \int_0^\infty e^{-x} dx = [-e^{-x}]_0^\infty = 1$

(11) $\int_0^\infty x e^{-2x} dx = [-\frac{x}{2} e^{-2x}]_0^\infty + \frac{1}{2}\int_0^\infty e^{-2x} dx = [-\frac{1}{4} e^{-2x}]_0^\infty = \frac{1}{4}$

(12) $\int_0^\infty \frac{1}{x^2}\log(1+a^2x^2) dx$

$= [-\frac{1}{x}\log(1+a^2x^2)]_0^\infty + \int_0^\infty \frac{1}{x}\frac{2a^2x}{1+a^2x^2} dx$

$= \int_0^\infty \frac{2}{x^2+\frac{1}{a^2}} dx = [2a\tan^{-1} ax]_0^\infty$

$= 2a\frac{\pi}{2} = a\pi$

$\lim_{x\to 0}\frac{\log(1+a^2x^2)}{x}$

$= \lim_{x\to 0}\frac{\frac{2a^2x}{1+a^2x^2}}{1} = 0$

$\lim_{x\to\infty}\frac{\log(1+a^2x^2)}{x} = 0$

5.2 (1) $\int_0^1 \log(1+\sqrt{x}) dx \qquad \sqrt{x} = t \qquad x = t^2 \quad dx = 2t\,dt$

$= \int_0^1 2t\log(1+t)dt = [t^2\log(1+t)]_0^1 - \int_0^1 \frac{t^2}{1+t} dt$

$= \log 2 - \int_0^1 (t - 1 + \frac{1}{1+t}) dt = \log 2 - [\frac{t^2}{2} - t + \log|1+t|]_0^1$

$= \log 2 - (-\frac{1}{2} + \log 2) = \frac{1}{2}$

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5.2 (2) $\displaystyle\int_0^1 \frac{2x}{\sqrt{1-x^2}}dx = \left[-2\sqrt{1-x^2}\right]_0^1 = 2$

(3) $\displaystyle\int_0^2 \frac{2x}{\sqrt{2x-x^2}}dx = \int_0^2 2\sqrt{\frac{x}{2-x}}\,dx$

$\sqrt{\dfrac{x}{2-x}} = t$ とおく

$x(1+t^2) = 2t^2$

$x = \dfrac{2t^2}{1+t^2}$ $\quad dx = \dfrac{4t(t^2+1)-2t^3}{(1+t^2)^2}dt$

$= \dfrac{4t}{(1+t^2)^2}dt$

$\displaystyle = \int_0^\infty 2t\frac{4t}{(1+t^2)^2}dt = 8\int_0^\infty \frac{t^2}{(t^2+1)^2}dt$

$t = \tan\theta$

$dt = \sec^2\theta\, d\theta$

$\displaystyle = 8\int_0^{\frac{\pi}{2}} \frac{\tan^2\theta}{\sec^4\theta}\sec^2\theta\, d\theta$

$\displaystyle = 8\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta = 8\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 2\pi$

(4) $\displaystyle\int_0^2 \frac{1}{\sqrt{x(2-x)}}dx = \int_0^2 \frac{1}{x}\sqrt{\frac{x}{2-x}}\,dx = \int_0^\infty \frac{1+t^2}{2t^2}\cdot t\,\frac{4t}{(1+t^2)^2}dt$

$\displaystyle = 2\int_0^\infty \frac{1}{1+t^2}dt = 2\left[\tan^{-1}t\right]_0^\infty = \pi$

5.3 (1) $\displaystyle\int_{-1}^2 \frac{x}{\sqrt[3]{1-x^2}}dx = \int_{-1}^0 \frac{x}{\sqrt[3]{1-x^2}}dx + \int_0^1 \frac{x}{\sqrt[3]{1-x^2}}dx + \int_1^2 \frac{x}{\sqrt[3]{1-x^2}}dx$

$\displaystyle\left(\int_{-1}^0 \frac{x}{\sqrt[3]{1-x^2}}dx = \int_1^0 \frac{-t}{\sqrt[3]{1-t^2}}(-dt) = -\int_0^1 \frac{t}{\sqrt[3]{1-t^2}}dt \right)$

$\displaystyle = \int_1^2 \frac{x}{\sqrt[3]{1-x^2}}dx$

$1-x^2 = -t$

$x\,dx = \frac{1}{2}dt$

$\displaystyle = \int_0^3 \frac{1}{(-t)^{\frac{1}{3}}}\frac{1}{2}dt = -\frac{1}{2}\int_0^3 t^{-\frac{1}{3}}dt$

$\displaystyle = -\frac{1}{2}\left[\frac{3}{2}t^{\frac{2}{3}}\right]_0^3 = -\frac{3}{4}\cdot 3^{\frac{2}{3}} = -\frac{3}{4}\sqrt[3]{9}$

(2) $f(x) = \dfrac{x}{\sqrt[3]{(x^2-1)^2}}$ は奇関数

$\displaystyle\int_{-1}^3 \frac{x}{\sqrt[3]{(x^2-1)^2}}dx = \int_1^3 \frac{x}{\sqrt[3]{(x^2-1)^2}}dx$

$x^2-1 = t$

$x\,dx = \frac{1}{2}dt$

$\displaystyle = \int_0^8 t^{-\frac{2}{3}}\frac{1}{2}dt = \frac{1}{2}\frac{3}{1}\left[t^{\frac{1}{3}}\right]_0^8 = \frac{3}{2}\cdot 2 = 3$

5.4 $\displaystyle I_{MN} = \int_0^1 x^N(\log x)^M dx = \left[\frac{1}{N+1}x^{N+1}(\log x)^M\right]_0^1 - \frac{M}{N+1}\int_0^1 x^N(\log x)^{M-1}dx$

$\displaystyle = (-1)\frac{M}{N+1}\int_0^1 x^N(\log x)^{M-1}dx = (-1)^2\frac{M(M-1)}{(N+1)^2}\int_0^1 x^N(\log x)^{M-2}dx$

$\displaystyle = (-1)^M\frac{M!}{(N+1)^M}\int_0^1 x^N dx = (-1)^M\frac{M!}{(N+1)^{M+1}}$

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5.5 $\int_0^\infty e^{-x}|\sin x|dx$ $\qquad 2n\pi \leqq x \leqq (2n+1)\pi$ のとき $\sin x \geqq 0$

$\qquad (2n-1)\pi \leqq x \leqq 2n\pi$ のとき $\sin x \leqq 0$

$$\int e^{-x}\sin x\,dx = -e^{-x}\sin x + \int e^{-x}\cos x\,dx$$
$$= -e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\sin x\,dx$$

$$\therefore \int e^{-x}\sin x\,dx = -\frac{e^{-x}}{2}(\sin x + \cos x)$$

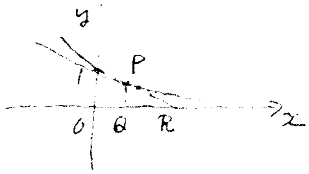
$$\therefore \int_{n\pi}^{(n+1)\pi} e^{-x}\sin x\,dx = -\frac{1}{2}\left[e^{-x}(\sin x + \cos x)\right]_{n\pi}^{(n+1)\pi} = -\frac{1}{2}\{(-1)^{n+1}e^{-(n+1)\pi} - (-1)^n e^{-n\pi}\}$$
$$= \frac{1}{2}\{(-1)^n e^{-n\pi} - (-1)^{n+1}e^{-(n+1)\pi}\} = \frac{(-1)^n}{2}\{e^{-n\pi} + e^{-(n+1)\pi}\}$$

$$\therefore \int_{n\pi}^{(n+1)\pi} e^{-x}|\sin x|\,dx = \frac{1}{2}(e^{-n\pi} + e^{-(n+1)\pi})$$

$$\int_0^\infty e^{-x}|\sin x|dx = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{e^{n\pi}} + \frac{1}{e^{(n+1)\pi}}\right) = \frac{1}{2}\left\{\frac{1}{1-e^{-\pi}} + \frac{e^{-\pi}}{1-e^{-\pi}}\right\}$$
$$= \frac{1+e^{-\pi}}{2(1-e^{-\pi})}$$

5.6 $y = e^{-ax}$ $\quad a > 0$

(1) グラフ



(2) 接線 $y' = -ae^{-ax}$

$Y - e^{-ax} = -ae^{-ax}(X - x)$

$Y = 0 \quad X = \frac{1}{a} + x$

$P(x, y) \quad Q(x, 0) \quad R\left(\frac{1}{a} + x, 0\right)$

$QR = \frac{1}{a}$

(3) $\int_0^\infty e^{-ax}dx = \left[-\frac{1}{a}e^{-ax}\right]_0^\infty = \frac{1}{a}$

$\int_0^\infty x e^{-ax}dx = \left[-\frac{x}{a}e^{-ax}\right]_0^\infty + \frac{1}{a}\int_0^\infty e^{-ax}dx = \frac{1}{a^2}$

5.7 $\quad I_n = \int_0^{\frac{\pi}{2}} \frac{\sin nx}{\sin x}dx$

(1) $I_1 = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x}dx = \frac{\pi}{2}$

(2) $I_{2n+1} - I_{2n-1} = \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x}dx$

$= \int_0^{\frac{\pi}{2}} \frac{2\cos 2nx \sin x}{\sin x}dx = \int_0^{\frac{\pi}{2}} 2\cos 2nx\,dx$

$= \left[\frac{2}{2n}\sin 2nx\right]_0^{\frac{\pi}{2}} = 0$

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5.7 (3) $I_{2n+2} - I_{2n} = \int_0^{\frac{\pi}{2}} \frac{\sin(2n+2)x - \sin 2nx}{\sin x} dx = \int_0^{\frac{\pi}{2}} \frac{2\cos(2n+1)x \sin x}{\sin x} dx$

$= \frac{2}{2n+1} \left[\sin(2n+1)x\right]_0^{\frac{\pi}{2}} = \frac{(-1)^n 2}{2n+1}$.

(4) $I_{2n-1} = I_{2n-3} \qquad = I_1 = \frac{\pi}{2} \quad \therefore\ I_{2n-1} = \frac{\pi}{2}$

5.8 $I_n = \int_0^{\pi} \frac{\sin nx}{\sin x} dx$ とおく

$I_{n+2} - I_n = \int_0^{\pi} \frac{\sin(n+2)x - \sin nx}{\sin x} dx = \int_0^{\pi} \frac{2\cos(n+1)x \sin x}{\sin x} dx$

$= \frac{2}{n+1} \left[\sin(n+1)x\right]_0^{\pi} = 0$

$\therefore\ I_{n+2} - I_n = 0 \qquad I_1 = \int_0^{\pi} \frac{\sin x}{\sin x} dx = \pi \quad I_2 = \int_0^{\pi} \frac{\sin 2x}{\sin x} dx = 2\int_0^{\pi} \cos x\, dx$

$= 2\left[\sin x\right]_0^{\pi} = 0$

$\therefore\ I_{2n} = 0$

$I_{2n-1} = \pi$

5.9 $f(x) \quad [0, 1]$

$\int_0^{\pi} f(\sin\theta) d\theta = \int_0^{\frac{\pi}{2}} f(\sin\theta) d\theta + \int_{\frac{\pi}{2}}^{\pi} f(\sin\theta) d\theta \qquad \pi - \theta = t$ とおく

$= \int_0^{\frac{\pi}{2}} f(\sin\theta) d\theta + \int_{\frac{\pi}{2}}^{\pi} f(\sin(\pi - \theta)) d\theta$

$= \int_0^{\frac{\pi}{2}} f(\sin\theta) d\theta + \int_{\frac{\pi}{2}}^{1} f(\sin t)(t\, dt)$

$= 2\int_0^{\frac{\pi}{2}} f(\sin\theta) d\theta \qquad \sin\theta = t$ とおく

$\theta = \sin^{-1} t$

$= 2\int_0^1 \frac{f(t)}{\sqrt{1-t^2}} dt \qquad d\theta = \frac{1}{\sqrt{1-t^2}} dt$

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6.1 $$\int_0^1 x^m(1-x)^n dx = \left[\frac{1}{m+1}x^{m+1}(1-x)^n\right]_0^1 + \frac{n}{m+1}\int_0^1 x^{m+1}(1-x)^{n-1}dx$$

$$= \frac{n}{m+1}\int_0^1 x^{m+1}(1-x)^{n-1} = \frac{n(n-1)}{(m+1)(m+2)}\int_0^1 x^{m+2}(1-x)^{n-2}dx$$

$$= \frac{n!}{(m+1)(m+2)\cdot(m+n)}\int_0^1 x^{m+n}dx$$

$$= \frac{n!}{(m+1)(m+2)\cdots(m+n)(m+n+1)} = \frac{n!\,m!}{(m+n+1)!}$$

6.2 $s>0$ $\Gamma(s)=\int_0^\infty e^{-x}x^{s-1}dx$ ガンマ関数

(1) $s>1$ のとき

$$\Gamma(s+1)=\int_0^\infty e^{-x}x^s dx = \left[-e^{-x}x^s\right]_0^\infty + s\int_0^\infty e^{-x}x^{s-1}dx$$

$$= s\int_0^\infty e^{-x}x^{s-1}dx = s\,\Gamma(s)$$

(2) $\int_0^\infty \alpha e^{-0.5t}t^{s-1}dt = 1$

$$\int_0^\infty \alpha\, e^{-0.5t}t^{s-1}dt$$ $0.5t = x$ $dt = 2dx$, $t = 2x$

$$= \int \alpha\, e^{-x}(2x)^{s-1}\cdot 2dx$$

$$= \alpha\cdot 2^s\int_0^\infty e^{-x}x^{s-1}dx = 2^s\alpha\,\Gamma(s)$$

$2^s\alpha\Gamma(s) = 1$ $\alpha = \frac{1}{2^s\Gamma(s)}$

6.3 $f(\theta) = \{\sin(n\theta - x\sin\theta)\}' = \cos\{n\theta - x\sin\theta\}(n - x\cos\theta)$

$$\therefore \int_0^\pi f(\theta)d\theta = \int_0^\pi \cos\{n\theta - x\sin\theta\}(n - x\cos\theta)d\theta$$

$$\left[\sin(n\theta - x\sin\theta)\right]_0^\pi = \int_0^\pi n\cos\{n\theta - x\sin\theta\}d\theta - \int_0^\pi x\cos\theta\cos\{n\theta - x\sin\theta\}d\theta$$

$$n\int_0^\pi \cos(n\theta - x\sin\theta)d\theta = \int_0^\pi x\cos\theta\cos(n\theta - x\sin\theta)d\theta$$

$$\therefore J_n(x) = \frac{x}{n\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\cos\theta\, d\theta$$

$$x\{J_{n+1}(x) + J_{n-1}(x)\} = \frac{x}{\pi}\int_0^\pi [\cos((n+1)\theta - x\sin\theta) + \cos((n-1)\theta - x\sin\theta)]d\theta$$

$$= \frac{x}{\pi}\int_0^\pi 2\cos(n\theta - x\sin\theta)\cos\theta\, d\theta$$

$$= 2nJ_n(x)$$

$$\therefore xJ_{n+1}(x) - 2nJ_n(x) - xJ_{n-1}(x) = 0$$

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6.4 $s > 0$ $\quad \Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx$

(1) $\Gamma(s+1) = \int_0^\infty e^{-x} x^s dx = \left[-e^{-x} x^s\right]_0^\infty + s\int_0^\infty e^{-x} x^{s-1} dx$

$= s\,\Gamma(s)$

(2) $\Gamma(1) = \int_0^\infty e^{-x} dx = \left[-e^{-x}\right]_0^\infty = 1$

(1より)
$\Gamma(5) = 4\,\Gamma(4) = 4\cdot 3\,\Gamma(3) = 4\cdot 3\cdot 2\,\Gamma(2) = 4\cdot 3\cdot 2\cdot 1\,\Gamma(1) = 4! = 24$

6.5 m, n 自然数 $\quad 0 \leq P \leq 1$

$$\int_0^P x^m (1-x)^n dx = \frac{1}{m+1}\left[x^{m+1}(1-x)^n\right]_0^P + \frac{n}{m+1}\int_0^P x^{m+1}(1-x)^{n-1} dx$$

$$= \frac{1}{m+1} P^{m+1}(1-P)^n + \frac{n}{(m+1)(m+2)}\left[x^{m+2}(1-x)^{n-1}\right]_0^P + \frac{n(n-1)}{(m+1)(m+2)}\int_0^P x^{m+2}(1-x)^{n-2} dx$$

$$= \frac{1}{m+1} P^{m+1}(1-P)^n + \frac{n}{(m+1)(m+2)} P^{m+2}(1-P)^{n-1} + \frac{n(n-1)}{(m+1)(m+2)(m+3)} P^{m+3}(1-P)^{n-2}$$

$$+ \frac{n!}{(m+1)(m+2)\cdots(m+n)}\int_0^P x^{m+n} dx$$

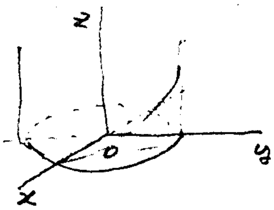
$$(m+1)\int_0^P x^m (1-x)^n dx = P^{m+1}(1-P)^n + \frac{n}{m+2} P^{m+2}(1-P)^{n-1} + \cdots + \frac{n!}{(m+2)\cdots(m+n+1)} P^{m+n+1}$$

$$= P^{m+1}\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{m+k+1}{k}} P^k (1-P)^{n-k}$$

$$P^{m+1}\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{m+k+1}{k}} P^k (1-P)^{n-k} = (m+1)\int_0^P x^m (1-x)^n dx$$

P.51

§7 積分の応用（最大・最小）



7.1 $x = a\cos t \quad y = a\sin t, \quad z = \frac{h}{2\pi}t$

$\pi a h = C$

(1). $0 \leqq t \leqq 2\pi$

弧の長さ $\int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$\therefore l = \int_0^{2\pi} \sqrt{a^2 + \frac{h^2}{4\pi^2}}\, dt = \frac{\sqrt{4\pi^2 a^2 + h^2}}{2\pi} 2\pi$

$\therefore l = \sqrt{4\pi^2 a^2 + h^2}$

$\frac{dl}{dh} = \frac{1}{2\sqrt{4\pi^2 a^2 + h^2}} \left(8\pi^2 a \frac{da}{dh} + 2h\right) = 0$

$\therefore 4\pi^2 a \frac{da}{dh} + h = 0$. $\quad \pi a h = C$ より $\pi h \frac{da}{dh} + \pi a = 0$

$\therefore \frac{da}{dh} = -\frac{a}{h}$

$\therefore 4\pi^2 a\left(-\frac{a}{h}\right) + h = 0$

$-4\pi^2 a^2 + h^2 = 0 \quad h = 2\pi a \quad \therefore 2\pi^2 a^2 = C$

$\therefore l = \sqrt{4\pi^2 a^2 + 4\pi^2 a^2} = 2\sqrt{2\pi^2 a^2} = 2\sqrt{C}$

(2) (1)より $h = 2\pi a$

7.2 $I = \int_0^{2\pi} (x\cos\theta + y\sin\theta - \theta)^2\, d\theta$

$= \int_0^{2\pi} \{x^2\cos^2\theta + y^2\sin^2\theta + \theta^2 + 2xy\sin\theta\cos\theta - 2x\theta\cos\theta - 2y\theta\sin\theta\}\, d\theta$

$= \pi x^2 + \pi y^2 + \frac{8}{3}\pi^3 - \left[2\theta(x\sin\theta - y\cos\theta)\right]_0^{2\pi} + \int_0^{2\pi} 2(x\sin\theta - y\cos\theta)\, d\theta$

$= \pi x^2 + \pi y^2 + \frac{8}{3}\pi^3 - (4\pi(-y)) + 2\left[-x\cos\theta - y\sin\theta\right]_0^{2\pi}$

$= \pi x^2 + \pi y^2 + \frac{8}{3}\pi^3 + 4\pi y$

$\frac{\partial I}{\partial x} = 2\pi x = 0 \qquad x = 0$

$\frac{\partial I}{\partial y} = 2\pi y + 4\pi = 0 \quad y = -2$

$\therefore (x, y) = (0, -2)$ のとき 最小値 $I = 4\pi + \frac{8}{3}\pi^3 - 8\pi$

$= \frac{8}{3}\pi^3 - 4\pi = \frac{4\pi(2\pi^2 - 3)}{3}$

7.3 $\int e^{-t}\sin t\, dt = -e^{-t}\sin t + \int e^{-t}\cos t\, dt$

$= -e^{-t}\sin t - e^{-t}\cos t - \int e^{-t}\sin t\, dt$

$\therefore \int e^{-t}\sin t\, dt = \frac{-e^{-t}}{2}(\sin t + \cos t)$

$G(x) = \int_0^{x^2} e^{-t}\sin t\, dt = -\frac{e^{-x^2}}{2}(\sin x^2 + \cos x^2) + \frac{1}{2}$

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7.3 $G(x) = \frac{1}{2} - \frac{\sqrt{2}}{2} e^{-x^2} \sin\left(x^2 + \frac{\pi}{4}\right)$

$G'(x) = \sqrt{2}\, x e^{-x^2} \sin\left(x^2 + \frac{\pi}{4}\right) - \sqrt{2}\, x e^{-x^2} \cos\left(x^2 + \frac{\pi}{4}\right)$

$= 2x e^{-x^2} \sin x^2$

$\therefore G'(x) = 0$ のとき $x = 0,\ \pm\sqrt{n\pi}$

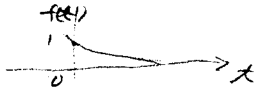
$G(0) = 0$ $\quad G(\sqrt{n\pi}) = \frac{1}{2} - \frac{\sqrt{2}}{2} e^{-n\pi} \sin\left(n\pi + \frac{\pi}{4}\right)$

$= \frac{1}{2} - \frac{\sqrt{2}}{2} e^{-n\pi} (-1)^n \frac{1}{\sqrt{2}}$

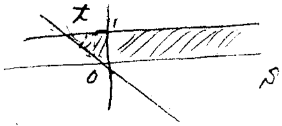
$= \frac{1 - (-1)^n e^{-n\pi}}{2}$

$\therefore$ 最小値 0 最大値 $\frac{1 + e^{-\pi}}{2}$ ($n = 1$ のとき)

7.4 $f(t) = \begin{cases} e^{-t} & (t \geq 0) \\ 0 & (t < 0) \end{cases}$



(1) $t + s \geq 0,\quad 0 < t < 1$



(2) $0 < t < 1$

$s < -1$ のとき $s + t < 1 - 1 = 0$ $\quad\therefore\ f(s+t) = 0$

$\therefore\ g(s) = \int_0^1 e^{2t} f(s+t)\, dt = 0$

$g(s) = \begin{cases} 0 & s < -1 \\ e^{-s}(e - e^{-s}) & -1 < s < 0 \\ e^{-s}(e - 1) & s > 0 \end{cases}$

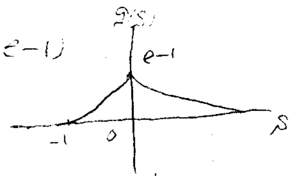
$-1 < s < 0$

$g(s) = \int_0^1 e^{2t} f(s+t)\, dt = \int_{-s}^1 e^{2t} e^{-s-t}\, dt$

$= \int_{-s}^1 e^{-s} e^{t}\, dt = e^{-s} \left[e^{t}\right]_{-s}^1 = e^{-s}(e - e^{-s})$

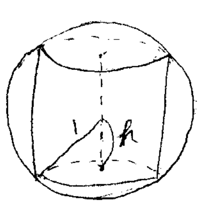
$0 \leq s$

$g(s) = \int_0^1 e^{2t} e^{-s-t}\, dt = e^{-s}\left[e^{t}\right]_0^1 = e^{-s}(e - 1)$



(3) $g(s)$ の最大値 $g(0) = e - 1$

7.5



$r^2 = 1 - h^2$

$V = \pi r^2 2h = 2\pi (1 - h^2) h$

$= 2\pi (h - h^3)$

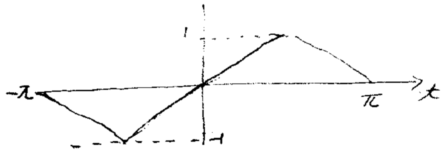
$\frac{dV}{dh} = 2\pi (1 - 3h^2) \qquad h = \frac{1}{\sqrt{3}}$

$V = 2\pi \frac{1}{\sqrt{3}} \left(1 - \frac{1}{3}\right) = \frac{4\pi}{3\sqrt{3}}$

P.51 　　§8 積分の応用（図形）

8.1

$$f(t)=\begin{cases}-\frac{2}{\pi}t-2 & -\pi\le t\le -\frac{\pi}{2}\\ \frac{2}{\pi}t & -\frac{\pi}{2}\le t\le \frac{\pi}{2}\\ -\frac{2}{\pi}t+2 & \frac{\pi}{2}\le t\le \pi\end{cases}$$



$$g(t)=a\sin t+b\sin 2t$$

$$I=\int_{-\pi}^{\pi}\{f(t)-g(t)\}^2dt$$

$$=\int_{-\pi}^{-\frac{\pi}{2}}\{-\frac{2}{\pi}t-2-a\sin t-b\sin 2t\}^2dt+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{\frac{2}{\pi}t-a\sin t-b\sin 2t\}^2dt$$

$$+\int_{\frac{\pi}{2}}^{\pi}\{-\frac{2}{\pi}t+2-a\sin t-b\sin 2t\}^2dt$$

$$\int_{-\pi}^{-\frac{\pi}{2}}\{-\frac{2}{\pi}t-2-a\sin t-b\sin 2t\}^2dt=\int_{\frac{\pi}{2}}^{\pi}\{\frac{2}{\pi}\tau-2+a\sin\tau+b\sin 2\tau\}^2d\tau \quad (t=-\tau)$$

$$\therefore\ I=2\left\{\int_{\frac{\pi}{2}}^{\pi}(\frac{2}{\pi}t-2+a\sin t+b\sin 2t)^2dt+\int_0^{\frac{\pi}{2}}(\frac{2}{\pi}t-a\sin t-b\sin 2t)^2dt\right\}$$

$$\int_0^{\frac{\pi}{2}}(\frac{2}{\pi}t-a\sin t-b\sin 2t)^2dt$$

$$=\int_0^{\frac{\pi}{2}}\{\frac{4}{\pi^2}t^2+a^2\sin^2t+b^2\sin^2 2t+2ab\sin t\sin 2t-\frac{4}{\pi}t(a\sin t+b\sin 2t)\}dt$$

$$=\frac{\pi}{3\cdot 2}+\frac{a^2\pi}{4}+\frac{b^2\pi}{4}+\frac{4}{3}ab-\frac{4b}{\pi}-\frac{4a}{\pi}$$

$$\int_{\frac{\pi}{2}}^{\pi}\{\frac{2}{\pi}t-2+a\sin t+b\sin 2t\}^2dt$$

$$=\int_{\frac{\pi}{2}}^{\pi}\{(\frac{2}{\pi}t-2)^2+a^2\sin^2t+b^2\sin^2 2t+2ab\sin t\sin 2t+4(\frac{t}{\pi}-1)(a\sin t+b\sin 2t)\}dt$$

$$=\frac{\pi}{6}+\frac{a^2}{4}\pi+\frac{b^2}{4}\pi-\frac{4}{3}ab+\frac{4b}{\pi}-\frac{4}{\pi}a$$

$$I=2\pi\left(\frac{1}{3}+\frac{a^2}{2}+\frac{b^2}{2}\right)-\frac{16}{\pi}a \qquad \frac{\partial I}{\partial b}=2\pi b=0 \quad b=0$$

$$\frac{\partial I}{\partial a}=2\pi a-\frac{16}{\pi} \quad a=\frac{8}{\pi^2}$$

$a=\frac{8}{\pi^2}$, $b=0$ のとき最小

$$\int_{-\pi}^{\pi}\{f(t)\}^2dt=4\int_0^{\frac{\pi}{2}}(\frac{2}{\pi}t)^2dt=\frac{16}{\pi^2}\left[\frac{t^3}{3}\right]_0^{\frac{\pi}{2}}=\frac{2}{3}\pi$$

$$E=\frac{\frac{2}{3}\pi-\frac{64}{\pi^3}}{\frac{2}{3}\pi}=1-\frac{96}{\pi^4}$$

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8.2 $f(x) = e^{-x}\sin x$

(1) $\int e^{-x}\sin x\,dx = \frac{-1}{2}e^{-x}(\sin x + \cos x)$

$$S_k = \left| -\frac{1}{2}\left[e^{-x}(\sin x + \cos x)\right]_{k\pi}^{(k+1)\pi} \right|$$

$$= \left| -\frac{1}{2}e^{-(k+1)\pi}(-1)^{k+1} + \frac{1}{2}e^{-k\pi}(-1)^k \right| = \frac{1}{2}\left(e^{-k\pi} + e^{-(k+1)\pi}\right)$$

$$= \frac{e^{-k\pi}}{2}(1 + e^{-\pi})$$

(2) $\sum_{k=0}^{n} S_k = \frac{1+e^{-\pi}}{2} \cdot \frac{1-e^{-(n+1)\pi}}{1-e^{-\pi}}$

(3) $\lim_{n\to\infty}\sum_{k=0}^{n} S_k = \frac{1+e^{-\pi}}{2(1-e^{-\pi})}$

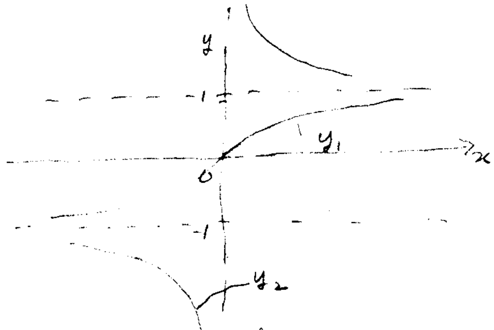
(4) $\frac{1+e^{-\pi}}{2(1-e^{-\pi})}\left(1 - \frac{1}{100}\right) \leqq \frac{(1+e^{-\pi})(1-e^{-(n+1)\pi})}{2(1-e^{-\pi})}$

$\frac{1}{100} \geqq e^{-(n+1)\pi}$ $\quad (n+1)\pi \geqq \log 100 \quad n+1 \geqq \frac{1}{\pi}\log 100$

$\therefore n = 1$

8.3 $y_1 = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh x$

$y_2 = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \coth x$



(1) $a > 1$ のとき

$$\int_1^a (y_2 - y_1)\,dx = \left[\log|e^x - e^{-x}| - \log|e^x + e^{-x}|\right]_1^a$$

$$= \left[\log(\tanh x)\right]_1^a = \log(\tanh a) - \log(\tanh 1)$$

$0 < a < 1$ のとき

$$\int_a^1 (y_1 - y_2)\,dx = \log(\tanh 1) - \log(\tanh a)$$

$a < 0$ のとき なし

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8.4 $x=a(\theta-\sin\theta)$ $y=a(1-\cos\theta)$

$$S=\int_0^{2\pi a} y\,dx=\int_0^{2\pi} a(1-\cos\theta)\,a(1-\cos\theta)\,d\theta=2a^2\int_0^{\pi}(1-\cos\theta)^2\,d\theta$$

$$=2a^2\int_0^{\pi}\left(2\sin^2\frac{\theta}{2}\right)^2 d\theta=8a^2\int_0^{\pi}\sin^4\frac{\theta}{2}\,d\theta$$

$\frac{\theta}{2}=t$
$d\theta=2dt$

$$=16a^2\int_0^{\frac{\pi}{2}}\sin^4 t\,dt=16a^2\,\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}=3a^2\pi$$

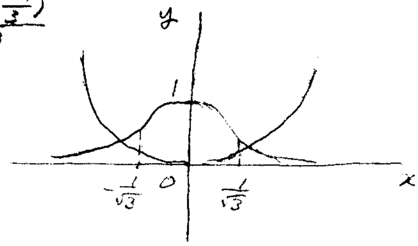
8.5 (a) $y=\frac{1}{x^2+1}$ $y'=\frac{-2x}{(x^2+1)^2}$ $y''=\frac{6(x^2-\frac{1}{3})}{(x^2+1)^3}$

$x\to\pm\infty$ のとき $y\to0$

∴ 漸近線 $y=0$

変曲点 $\left(\frac{1}{\sqrt3},\frac{3}{4}\right)$ $\left(-\frac{1}{\sqrt3},\frac{3}{4}\right)$

極大値 $f(0)=1$



(b) $\frac{1}{x^2+1}=\frac{x^2}{2}$ $2=x^4+x^2$

$x^4+x^2-2=0$ $(x^2+2)(x^2-1)=0$ $x=\pm1$

$$\int_{-1}^{1}\left(\frac{1}{x^2+1}-\frac{x^2}{2}\right)dx=\left[\tan^{-1}x-\frac{x^3}{6}\right]_{-1}^{1}=\frac{\pi}{2}-\frac{1}{3}$$

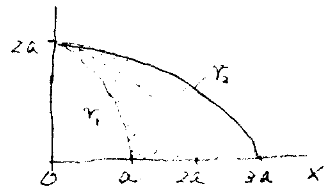
8.6 (1) $0\le\xi\le\frac{\pi}{2}$, $-1\le\eta\le1$

$$r=\left\{\frac{a(1-\eta)}{2}+\frac{3a(1+\eta)}{2}-2a\right\}\cos\xi+2a$$

$$=a\eta\cos\xi+2a$$

$\theta=\xi$

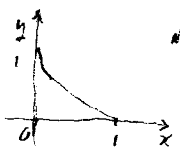
$r_1=-a\cos\theta+2a$ $r_2=a\cos\theta+2a$



(2) $S_1=\frac{1}{2}\int_0^{\frac{\pi}{2}}(-a\cos\theta+2a)^2d\theta$ $S_2=\frac{1}{2}\int_0^{\frac{\pi}{2}}(a\cos\theta+2a)^2d\theta$

$$S_2-S_1=\int_0^{\frac{\pi}{2}}4a^2\cos\theta\,d\theta=4a^2\left[\sin\theta\right]_0^{\frac{\pi}{2}}=4a^2$$

8.7



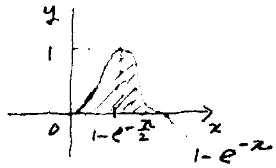
$\sqrt{x}+\sqrt{y}=1$ $y=1+x-2\sqrt{x}$

$$S=\int_0^1(1+x-2\sqrt{x})\,dx=\left[x+\frac{x^2}{2}-\frac{4}{3}x^{\frac{3}{2}}\right]_0^1=\frac{1}{6}$$

P.52

8.8 $\quad 0 \leqq t \leqq \pi$

(1) $\quad x = 1 - e^{-t}, \quad y = \sin^{2n} t$



t	0	$\frac{\pi}{2}$	π
x	0	$1-e^{-\frac{\pi}{2}}$	$1-e^{-\pi}$
y	0	1	0

$$S_n = \int_0^{\pi} \sin^{2n} t \; e^{-t} dt = \int_0^{\pi} e^{-t} \sin^{2n-2} t \,(1 - \cos^2 t)\, dt$$

$$= S_{n-1} - \int_0^{\pi} e^{-t} \sin^{2n-2} t \cos t \cos t \, dt$$

$$= S_{n-1} - \left\{ \frac{1}{2n-1} \left[e^{-t} \sin^{2n-1} t \cos t \right]_0^{\pi} - \frac{1}{2n-1} \int e^{-t} (-\cos t - \sin t) \sin^{2n-1} t \, dt \right\}$$

$$= S_{n-1} - \frac{1}{2n-1} \int_0^{\pi} e^{-t} \sin^{2n-1} t \cos t \, dt - \frac{1}{2n-1} \int_0^{\pi} e^{-t} \sin^{2n} t \, dt$$

$$= S_{n-1} - \frac{1}{2n-1} S_n - \frac{1}{2n(2n-1)} \left[e^{-t} \sin^{2n} x \right]_0^{\pi} - \frac{1}{2n(2n-1)} \int_0^{\pi} e^{-t} \sin^{2n} t \, dt$$

$$\therefore S_n = S_{n-1} - \left(\frac{1}{2n-1} + \frac{1}{2n(2n-1)} \right) S_n$$

$$\left(1 + \frac{2n+1}{2n(2n-1)}\right) S_n = S_{n-1} \qquad S_n = \frac{2n(2n-1)}{4n^2+1} S_{n-1}$$

(2) $\quad S_n = \frac{2n(2n-1)}{4n^2+1} S_{n-1}$

$$\therefore S_n = \frac{2n(2n-1)}{4n^2+1} \cdot \frac{(2n-2)(2n-3)}{4(n-1)^2+1} \cdots \frac{2 \cdot 1}{4+1} S_0 = \frac{(2n)!}{\prod_{k=1}^{n} (4k^2+1)} S_0$$

$$S_0 = \int_0^{\pi} e^{-t} dt = \left[-e^{-t} \right]_0^{\pi} = 1 - e^{-\pi}$$

$$\therefore S_n = \frac{(2n)!}{\prod_{k=1}^{n} (4k^2+1)} (1 - e^{-\pi}) \qquad \left(\prod_{k=1}^{5} a_k = a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5 \right)$$

8.9 $\quad 2x^2 + 2xy + y^2 = 1$

(1) $\quad 4x + 2y + (2x + 2y) y' = 0 \qquad y' = \frac{-(2x+y)}{x+y}$

$$4 + 2y' + (2 + 2y') y' + (2x + 2y) y'' = 0$$

$$2 - 2\frac{2x+y}{x+y} + \frac{(2x+y)^2}{(x+y)^2} + (x+y) y'' = 0$$

$$\frac{2x^2 + 2xy + y^2}{(x+y)^2} + (x+y) y'' = 0, \quad y'' = -\frac{1}{(x+y)^3}$$

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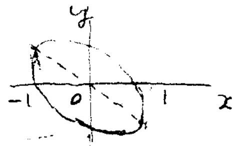
(2) $2x^2+2xy+y^2-1=0$



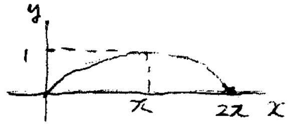
$y=-x\pm\sqrt{x^2-(2x^2-1)}=-x\pm\sqrt{1-x^2}$ $\qquad |x|\le 1$

$S=\int_{-1}^{1}\{-x+\sqrt{1-x^2}-(-x-\sqrt{1-x^2})\}dx$

$=2\int_{-1}^{1}\sqrt{1-x^2}\,dx \qquad x=\sin t \quad dx=\cos t\,dt$

$=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt=2\cdot 2\,\frac{\pi}{4}=\pi$

8.10 $x=t-\sin t$
$y=1-\cos t$ $\qquad$ サイクロイド



$S=\int_0^{2\pi}(1-\cos t)^2dt=\int_0^{2\pi}(2\cos^2\frac{t}{2})^2dt$

$=4\int_0^{2\pi}\cos^4\frac{t}{2}dt=8\cdot 2\int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta \qquad \frac{t}{2}=\theta$

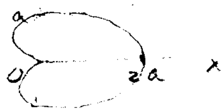
$=16\cdot\frac{3}{4}\ \frac{1}{2}\cdot\frac{\pi}{2}=3\pi$

8.11 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad y=\pm\frac{b}{a}\sqrt{a^2-x^2}$

$S=2\frac{b}{a}\int_{-a}^{a}\sqrt{a^2-x^2}\,dx \qquad x=a\sin t$

$=2\frac{b}{a}a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt=\pi ab$

8.12 $r=a(1+\cos\theta)$ $\qquad$ カージオイド

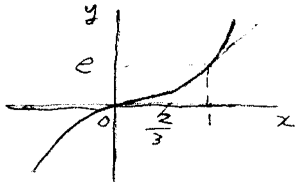


$S=\frac{1}{2}\int_0^{2\pi}a^2(1+\cos\theta)^2d\theta$

$=\frac{1}{2}\int_0^{2\pi}a^2(2\sin^2\frac{\theta}{2})^2d\theta=2a^2\int_0^{2\pi}\sin^4\frac{\theta}{2}\,d\theta \qquad \frac{\theta}{2}=t$

$=4a^2\int_0^{\pi}\sin^4 t\,dt=8a^2\cdot\frac{3}{4}\cdot\frac{1}{2}\frac{\pi}{2}=\frac{3}{2}a^2\pi$

8.13 $y=xe^{x^2} \qquad y'=(1+2x^2)e^{x^2} \qquad (1,e)$



(1) $y=3e(x-1)+e \qquad y=3ex-2e$

(2) $S=\int_0^{\frac{2}{3}}xe^{x^2}+\int_{\frac{2}{3}}^{1}\{xe^{x^2}-(3ex-2e)\}dx$

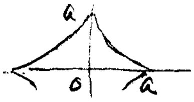
$=\left[\frac{1}{2}e^{x^2}\right]_0^1-\left[\frac{3}{2}x^2-2x\right]_{\frac{2}{3}}^{1}e=\frac{e}{2}-\frac{1}{2}-\frac{e}{6}=\frac{e}{3}-\frac{1}{2}$

P.52

§9 積分の応用(長さ)

9.1 $x = a\cos^3 t$ $\qquad \frac{dx}{dt} = -3a\cos^2 t\sin t$

$y = a\sin^3 t$ $\qquad \frac{dy}{dt} = 3a\sin^2 t\cos t$

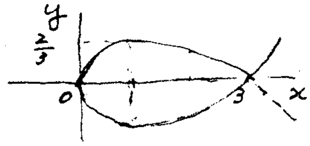


$$\int_0^{\frac{\pi}{2}} \sqrt{9a^2\sin^2 t\cos^2 t(\cos^2 t+\sin^2 t)}\,dt = 3a\int_0^{\frac{\pi}{2}}\sin t\cos t\,dt = \frac{3a}{2}\left[\sin^2 t\right]_0^{\frac{\pi}{2}}$$

$$= \frac{3}{2}a \qquad \therefore\ 4\times\frac{3}{2}a = 6a$$

9.2 $x = t^2,\quad y = t - \frac{t^3}{3}$

t	0	1	$\sqrt{3}$
x	0	1	3
y	0	$\frac{2}{3}$	0



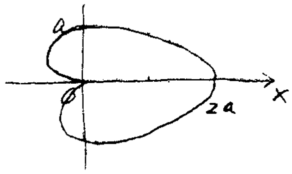
(1) $y' = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{1-t^2}{2t}$

$y'' = \dfrac{-2t^2-(1-t^2)}{2t^2}\Big/\dfrac{1}{2t} = \dfrac{-t^2-1}{4t^3}$

(2) $\frac{dx}{dt} = 2t \quad \frac{dy}{dt} = 1-t^2$

$$l = 2\int_0^{\sqrt{3}}\sqrt{4t^2+(1-t^2)^2}\,dt = 2\int_0^{\sqrt{3}}(t^2+1)\,dt = 2\left[\frac{t^3}{3}+t\right]_0^{\sqrt{3}} = 4\sqrt{3}$$

9.3 $r = a(1+\cos\theta) \quad a>0$



(2) $S = \displaystyle\int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta$

$$= 4a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\,d\theta = 8a^2\int_0^{\frac{\pi}{2}}\cos^4 t\,dt = 8a^2\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\pi = \frac{3}{2}a^2\pi$$

(3) $l = 2\displaystyle\int_0^{\pi}\sqrt{r^2+r'^2}\,d\theta = 2\int_0^{\pi}\sqrt{a^2(2+2\cos\theta)}\,d\theta = 2a\int_0^{\pi}2\cos\frac{\theta}{2}\,d\theta \qquad \frac{\theta}{2} = t$

$$= 8a\int_0^{\frac{\pi}{2}}\cos t\,dt = 8a\left[\sin t\right]_0^{\frac{\pi}{2}} = 8a$$

9.4 $x = a\cos^3\theta \quad y = a\sin^3\theta \quad a > 0 \quad 0 \le \theta \le \frac{\pi}{2}$

(1) $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{a}\right)^{\frac{2}{3}} = 1$

(2) $l = \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

$= \int_0^{\frac{\pi}{2}} \sqrt{9a^2 \sin^2\theta \cos^2\theta}\, d\theta = 3a \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta = 3a \left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{3}{2}a$

(3) $-\int_0^{\frac{\pi}{2}} a\sin^3 t\,(-3a)\cos^2\theta\sin\theta\, d\theta$

$= 3a^2 \int_0^{\frac{\pi}{2}} \sin^4\theta \cos^2\theta\, d\theta = 3a^2 \int_0^{\frac{\pi}{2}} (\sin^4\theta - \sin^6\theta)\, d\theta$

$= 3a^2 \left(\frac{3}{4}\frac{\pi}{4} - \frac{5}{6}\frac{3}{4}\frac{\pi}{4} \right) = \frac{9a^2}{16}\pi\, \frac{1}{6} = \frac{3a^2\pi}{32}$

(4) $x = \frac{a}{2} \quad \therefore \frac{1}{2} = \cos^3\theta \quad \cos\theta = \frac{1}{\sqrt[3]{2}} \quad \sin\theta = \sqrt{1 - \left(\frac{1}{2}\right)^{\frac{2}{3}}}$

$\frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta = \frac{1}{\sqrt{1 - \left(\frac{1}{2}\right)^{\frac{2}{3}}}}$

$y = \frac{1}{\sqrt{1 - \left(\frac{1}{2}\right)^{\frac{2}{3}}}} \left(x - \frac{a}{2}\right) + a\cdot\left(\sqrt{1 - \left(\frac{1}{2}\right)^{\frac{2}{3}}}\right)^3$

$\sqrt{1 - \left(\frac{1}{2}\right)^{\frac{2}{3}}}\, y = x - \frac{a}{2} + a\left(1 - \left(\frac{1}{2}\right)^{\frac{2}{3}}\right)^2$

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§.10 積分の応用（回転体）

10.1 $x=a(1-\sin t)$ $y=a(1-\cos t)$ $0\leqq t\leqq 2\pi$

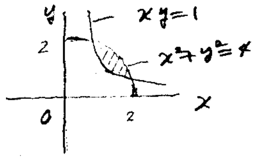
$V=\int_0^{2\pi}\pi y^2dx=\pi\int_0^{2\pi}a^2(1-\cos t)^2a(1-\cos t)dt$

$=a^3\pi\int_0^{2\pi}(2\sin^2\frac{t}{2})^3dt=8a^3\pi\int_0^{2\pi}\sin^6\frac{t}{2}dt$ $\frac{t}{2}=\theta$

$=16a^3\pi\int_0^{\pi}\sin^6 t\,dt=32a^3\pi\,\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}=5a^3\pi^2$

10.2 $xy=1$

$x^2+y^2=4$ $(x+y)^2=x^2+y^2+2xy=6$



∴ $x+y=\pm\sqrt{6}$ $x>0,\ y>0$ より

$x+y=\sqrt{6}$ $xy=1$

∴ $t^2-\sqrt{6}t+1=0$ の二根が x, y $t=\frac{\sqrt{6}\pm\sqrt{2}}{2}$ $\alpha=\frac{\sqrt{6}-\sqrt{2}}{2}$, $\beta=\frac{\sqrt{6}+\sqrt{2}}{2}$ とおく

∴ $V=\pi\int_\alpha^\beta(4-x^2-\frac{1}{x^2})dx=\pi\left[4x-\frac{x^3}{3}+\frac{1}{x}\right]_\alpha^\beta$

$=\pi\{4(\beta-\alpha)-\frac{1}{3}(\beta-\alpha)(\beta^2+\beta\alpha+\alpha^2)+\frac{\alpha-\beta}{\beta\alpha}\}=\pi\{4\sqrt{2}-\frac{\sqrt{2}}{3}(6-1)+\frac{\sqrt{2}}{1}\}=\frac{10\sqrt{2}}{3}\pi$

10.3 (1)

$x=rt-r\sin t$
$=r(t-\sin t)$
$y=r-r\cos t$
$=r(1-\cos t)$

∴ $x=r(t-\sin t)$
$y=r(1-\cos t)$

(2) $l=\int_0^{2\pi}\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt=\int_0^{2\pi}\sqrt{r^2(2-2\cos t)}\,dt=r\int_0^{2\pi}2\sin\frac{t}{2}dt=4r\left[-\cos\frac{t}{2}\right]_0^{2\pi}$

$=8r$

(3) $V=\pi\int_0^{2\pi}y^2dx=\pi\int_0^{2\pi}r^3(1-\cos t)^3dt=8\pi r^3\int_0^{2\pi}\sin^6\frac{t}{2}dt$ $\frac{t}{2}=\theta$

$=16\pi r^3\int_0^{\pi}\sin^6\theta\,d\theta=32\pi r^3\,\frac{5}{6}\,\frac{3}{4}\cdot\frac{\pi}{4}=5\pi^2r^3$

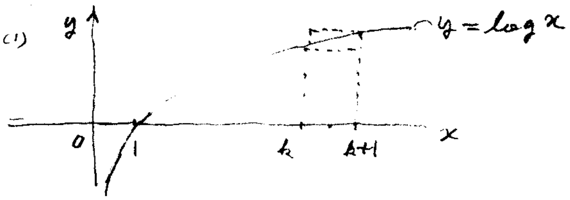
P.53

§11 区分求積法

11.1 $\displaystyle \lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} \sqrt{n^2-k^2} = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1-\left(\frac{k}{n}\right)^2}\,\frac{1}{n} = \int_0^1 \sqrt{1-x^2}\,dx$

$\displaystyle = \frac{1}{2}\left[x\sqrt{1-x^2} + \sin^{-1} x\right]_0^1 = \frac{1}{2}\,\frac{\pi}{2} = \frac{\pi}{4}$

11.2 (1)



$k \leqq x \leqq k+1$ のとき $\log k \leqq \log x \leqq \log(k+1)$ $k=1, 2, \cdots n-1$

$\displaystyle \therefore \int_k^{k+1} \log k\,dx < \int_k^{k+1} \log x\,dx < \int_k^{k+1} \log(k+1)\,dx$

$\displaystyle \therefore \log k < \int_k^{k+1} \log x\,dx < \log(k+1)$

$\displaystyle \sum_{k=1}^{n-1} \log k < \sum_{k=1}^{n-1} \int_k^{k+1} \log x\,dx < \sum_{k=1}^{n-1} \log(k+1)$

$\displaystyle \log 2 + \log 3 + \cdots + \log(n-1) < \int_1^n \log x < \log 2 + \log 3 + \cdots + \log n$

(2) (1) より $\displaystyle \log(n-1)! < \int_1^n \log x\,dx < \log n!$

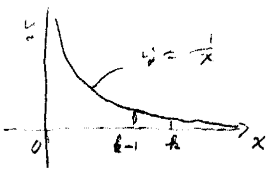
$\displaystyle \int_1^n \log x = \left[x\log x - x\right]_1^n = n\log n - n + 1$

$\therefore \log(n-1)! + \log n < (n+1)\log n - n + 1$ $\log n! < (n+1)\log n - n + 1$

又 $n\log n - n + 1 < \log n!$

$\therefore n\log n - n + 1 < \log n! < (n+1)\log n - n + 1$

11.3



$k-1 < x < k$ のとき $\displaystyle \frac{1}{k-1} > \frac{1}{x} > \frac{1}{k}$ $k = 2, 3, \cdots n$

$\displaystyle \int_{k-1}^{k} \frac{1}{k-1}\,dx > \int_{k-1}^{k} \frac{1}{x}\,dx > \int_{k-1}^{k} \frac{1}{k}\,dx$

$\displaystyle \therefore \frac{1}{k-1} > \left[\log x\right]_{k-1}^{k} > \frac{1}{k}$ $\displaystyle \frac{1}{k-1} > \log k - \log(k-1) > \frac{1}{k}$

$\displaystyle \therefore \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n-1} > \log n > \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

$\displaystyle 0 < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \log n < 1 - \frac{1}{n}$

$\displaystyle \therefore \frac{1}{n} < 1 + \frac{1}{2} + \cdots \frac{1}{n-1} + \frac{1}{n} - \log n < 1$ $\displaystyle 0 \leqq \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right) \leqq 1$

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11.4 $\lim_{n\to\infty}\sum_{i=1}^{n} f(\frac{i}{n})\frac{1}{n} = \int_0^1 f(x)\,dx$

$0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$　　$x_i - x_{i-1} = \Delta x_i$ とおくとき

$|\Delta| = \max\{\Delta x_1, \Delta x_2, \cdots \Delta x_n\}$

定義より

$\int_0^1 f(x)\,dx = \lim_{|\Delta|\to 0}\sum_{i=1}^{n} f(x_i)\Delta x_i$　　ここで $x_i = \frac{i}{n}$　$\Delta x_i = \frac{1}{n}$ とおくと

$= \lim_{n\to\infty}\sum_{i=1}^{n} f(\frac{i}{n})\frac{1}{n}$

(1) $\lim_{n\to\infty}\left\{\frac{n}{n^2+1} + \frac{n}{n^2+4} + \cdots + \frac{n}{n^2+n^2}\right\} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+(\frac{i}{n})^2}\frac{1}{n}$

$= \int_0^1 \frac{1}{1+x^2}\,dx = [\tan^{-1}x]_0^1 = \frac{\pi}{4}$

(2) $\lim_{n\to\infty}\left\{\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right\} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+\frac{i}{n}}\frac{1}{n}$

$= \int_0^1 \frac{1}{1+x}\,dx = [\log|1+x|]_0^1 = \log 2$

11.5 $a_n = \sqrt[n]{n!}/n$

$\frac{1}{n}\log\frac{1}{n} + \int_{\frac{1}{n}}^1 \log x\,dx < \log a_n < \int_{\frac{1}{n}}^1 \log x\,dx$

(1) $\log a_n = \frac{1}{n}\log n! - \log n = \frac{1}{n}\sum_{k=1}^{n}(\log k - \log n)$

$= \frac{1}{n}\sum_{k=1}^{n}\log\frac{k}{n}$

$\frac{k-1}{n} \leq x \leq \frac{k}{n}$　$(k = 2, 3, \cdots n)$ のとき

$\log\frac{k-1}{n} \leq \log x \leq \log\frac{k}{n}$　　$\int_{\frac{k-1}{n}}^{\frac{k}{n}} \log\frac{k-1}{n}\,dx < \int_{\frac{k-1}{n}}^{\frac{k}{n}} \log x\,dx < \int_{\frac{k-1}{n}}^{\frac{k}{n}} \log\frac{k}{n}\,dx$

$\frac{1}{n}\log\frac{k-1}{n} < \int_{\frac{k-1}{n}}^{\frac{k}{n}} \log x\,dx < \frac{1}{n}\log\frac{k}{n}$

$\therefore \frac{1}{n}\sum_{k=2}^{n}\log\frac{k-1}{n} < \int_{\frac{1}{n}}^1 \log x\,dx < \frac{1}{n}\sum_{k=2}^{n}\log\frac{k}{n}$

$\therefore \int_{\frac{1}{n}}^1 \log x\,dx + \frac{1}{n}\log\frac{1}{n} < \frac{1}{n}\sum_{k=1}^{n}\log\frac{k}{n} = \log a_n$　　①

$\frac{1}{n}\sum_{k=2}^{n}\log\frac{k-1}{n} + \frac{1}{n}\log\frac{n}{n} < \int_{\frac{1}{n}}^1 \log x\,dx + \frac{1}{n}\log\frac{n}{n}$　　$\log\frac{n}{n} = 0$ より

$\frac{1}{n}\sum_{k=1}^{n}\log\frac{k}{n} < \int_{\frac{1}{n}}^1 \log x\,dx$

$\therefore \log a_n < \int_{\frac{1}{n}}^1 \log x\,dx$　　②

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①②より $\frac{1}{n}\log\frac{1}{n} + \int_{\frac{1}{n}}^{1} \log x\,dx < \log a_n < \int_{\frac{1}{n}}^{1} \log x\,dx$

(2) $\int_{\frac{1}{n}}^{1} \log x\,dx = \left[x\log x - x\right]_{\frac{1}{n}}^{1} = -1 - \left(\frac{1}{n}\log\frac{1}{n} - \frac{1}{n}\right) = \frac{1}{n} - 1 - \frac{1}{n}\log\frac{1}{n}$

(1)より $\frac{1}{n}\log\frac{1}{n} + \frac{1}{n} - 1 - \frac{1}{n}\log\frac{1}{n} < \log a_n < \frac{1}{n} - 1 - \frac{1}{n}\log\frac{1}{n}$

$\frac{1}{n} - 1 < \log a_n < \frac{1}{n} - 1 - \frac{1}{n}\log\frac{1}{n}$ ---- ③

$\lim_{x\to 0} x\log x = \lim_{x\to 0}\frac{\log x}{\frac{1}{x}} = \lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0}(-x) = 0$ より

$-1 \le \lim_{n\to\infty}\log a_n \le -1$ ∴ $\lim_{n\to\infty}\log a_n = -1$

(3) (2)の③式より

$1-n < n\log a_n < 1-n-\log\frac{1}{n}$

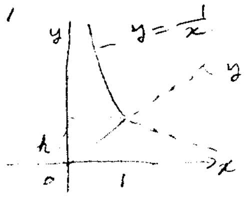
$1-n < \log a_n^n < 1-n-\log\frac{1}{n}$ $\quad e^{\log A} = A$

∴ $e^{1-n} < a_n^n < e^{1-n}n$

∴ $n^n e^{1-n} < n! < n^{n+1}e^{1-n}$

§12 積分の応用（物理）

12.1



$0 \le h \le 1$ のとき

$V(h) = \pi\int_0^h x^2\,dy = \frac{\pi h^3}{3}$

$1 \le h$ のとき

$V(h) = \frac{\pi}{3} + \pi\int_1^h \frac{1}{x^2}\,dx = \frac{\pi}{3} + \pi\left[-\frac{1}{x}\right]_1^h$

$= \left(\frac{4}{3} - \frac{1}{h}\right)\pi$

∴ $V(h) = \begin{cases} \frac{\pi}{3}h^3 & 0\le h\le 1 \\ \left(\frac{4}{3} - \frac{1}{h}\right)\pi & h>1 \end{cases}$

(1) $V = t$ $\quad t = \frac{\pi}{3}h^3 \quad h = \sqrt[3]{\frac{3t}{\pi}} \quad 0 \le t \le \frac{\pi}{3}$

$t = \left(\frac{4}{3} - \frac{1}{h}\right)\pi \quad h = \frac{3\pi}{4\pi - 3t} \quad \frac{\pi}{3} < t < \frac{4}{3}\pi$

(2) $0 \le t \le \frac{\pi}{3}$ のとき $h' = \sqrt[3]{\frac{3}{\pi}}\,\frac{1}{3}t^{-\frac{2}{3}} = \frac{1}{3}\sqrt[3]{\frac{3}{\pi t^2}}$

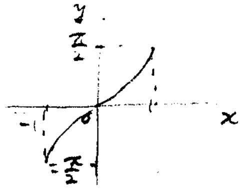
$\frac{\pi}{3} < t \le \frac{4}{3}\pi$ のとき $h' = \frac{9\pi}{(4\pi - 3t)^2}$

P.54

§13 積分の総合問題

13.1 $y=\sin^{-1}x \quad -\frac{\pi}{2}\leqq y\leqq\frac{\pi}{2}$

(1) グラフ



(2) $y=\sin^{-1}x$

$\therefore x=\sin y$

$\Delta x=\sin(y+\Delta y)-\sin y$ とおくと

$1=\dfrac{\sin(y+\Delta y)-\sin y}{\Delta x}=\dfrac{\Delta y}{\Delta x}\,\dfrac{\sin(y+\Delta y)-\sin y}{\Delta y}$

$\therefore 1=\lim_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}\cdot\dfrac{\sin(y+\Delta y)-\sin y}{\Delta y}=\lim_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}\,\dfrac{2\cos(y+\frac{\Delta y}{2})\sin(\frac{\Delta y}{2})}{\Delta y}$

$=\dfrac{dy}{dx}\cos y \qquad -\frac{\pi}{2}\leqq y\leqq\frac{\pi}{2}$ より $\cos y=\sqrt{1-\sin^2 y}$

$\therefore \dfrac{dy}{dx}=\dfrac{1}{\sqrt{1-x^2}}$

(3) $y'=\dfrac{1}{\sqrt{1-x^2}} \quad \therefore y''=\dfrac{x}{(\sqrt{1-x^2})^3}=\dfrac{x}{1-x^2}\cdot\dfrac{1}{\sqrt{1-x^2}}$

$\therefore (1-x^2)y''-xy'=0$ xで微分して

$(1-x^2)y'''-3xy''-y'=0$

(4) $\int\sin^{-1}x\,dx=x\sin^{-1}x-\int\dfrac{x}{\sqrt{1-x^2}}\,dx=x\sin^{-1}x+\sqrt{1-x^2}$

13.2 P.44, 14.1 と同じ

13.3 $f(x)=\begin{cases}1 & x=0\\ \dfrac{1}{x}\sin x & x>0\end{cases}$

$a_n=(-1)^{n-1}\displaystyle\int_{(n-1)\pi}^{n\pi}f(x)\,dx \qquad n=1,2,3\cdots$

(1) $(n-1)\pi<x<n\pi$ のとき

$0<(-1)^{n-1}\dfrac{\sin x}{x}<\dfrac{1}{x}<\dfrac{1}{(n-1)\pi}$

$\therefore 0<(-1)^{n-1}\displaystyle\int_{(n-1)\pi}^{n\pi}\dfrac{\sin x}{x}dx<\dfrac{1}{n-1}$

P.54

(2) $0 < t < \pi$ のとき　$\sin t = -\sin(\pi + t)$　$\dfrac{\sin t}{t} > \dfrac{-\sin(\pi+t)}{\pi+t}$

$(n-1)\pi < x < n\pi$ のとき　$\dfrac{(-1)^{n-1}\sin x}{x} > \dfrac{(-1)^n \sin(x+\pi)}{x+\pi}$

$$\therefore (-1)^{n-1}\int_{(n-1)\pi}^{n\pi}\frac{\sin x}{x}dx > (-1)^n\int_{(n-1)\pi}^{n\pi}\frac{\sin(x+\pi)}{x+\pi}dx$$　$x+\pi = t$ とおくと

$$= (-1)^n\int_{n\pi}^{(n+1)\pi}\frac{\sin t}{t}dt$$

$\therefore a_n > a_{n+1}$

(3) $\{a_n\}$は単調減少で $a_n > 0$

$a_n = (-1)^{n-1}\int_{(n-1)\pi}^{n\pi} f(x)\,dx$　$\therefore \int_{(n-1)\pi}^{n\pi} f(x)\,dx = (-1)^{n-1}a_n$

$$\therefore \int_0^{n\pi} f(x)\,dx = \int_0^{\pi}\frac{\sin x}{x}dx + \int_{\pi}^{2\pi}\frac{\sin x}{x}dx + \int_{2\pi}^{3\pi}\frac{\sin x}{x}dx + \cdots + \int_{(n-1)\pi}^{n\pi}\frac{\sin x}{x}dx$$

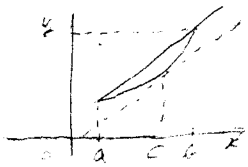
$$= a_1 - a_2 + a_3 - \cdots + (-1)^{n-1}a_n$$

$$\therefore \int_0^{\infty} f(x)\,dx = a_1 - a_2 + a_3 + \cdots + (-1)^{n-1}a_n + \cdots$$

これは交項級数で $a_n > a_{n+1}$　$\lim_{n\to\infty} a_n = 0$

$\therefore$ 収束する。

13.4　微分についての平均値の定理



$f(x)$ が $[a, b]$ で連続 (a, b) で微分可能なとき

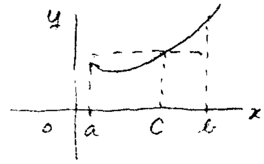
$\dfrac{f(b)-f(a)}{b-a} = f'(c)$ となる c $(a < c < b)$ がある。

$[a, b]$ における $f(x)$ の平均変化率に等しい微係数をもつ点 c $(a < c < b)$ がある

積分についての平均値の定理

$f(x)$ が $[a, b]$ で連続のとき　$\int_a^b f(x)\,dx = f(c)(b-a)$

を満たす c $(a < c < b)$ が存在する



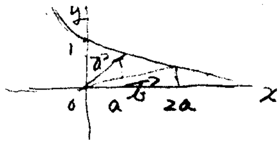
$S = \int_a^b f(x)\,dx$ を面積とするとき横 $(b-a)$ たて $f(c)$

の長方形の面積 $= S$ となる $f(c)$ $(a < c < b)$

が存在する

p 54

13.5 $y=\frac{1}{\sqrt{2x+1}}$



(1) $y=\frac{1}{\sqrt{2x+1}}$ で $x=a$

$\vec{a}=\left(a,\ \frac{1}{\sqrt{2a+1}}\right)$ $\vec{b}=\left(2a,\ \frac{1}{\sqrt{4a+1}}\right)$

$\vec{a}=a\vec{i}+\frac{1}{\sqrt{2a+1}}\vec{j}$ $\vec{b}=2a\vec{i}+\frac{1}{\sqrt{4a+1}}\vec{j}$

$\therefore\ \vec{a}\cdot\vec{b}=2a^2+\frac{1}{\sqrt{(2a+1)(4a+1)}}$

(2) $x=a$ における接線 $y'=-\frac{1}{2}(2x+1)^{-\frac{3}{2}}\cdot 2=\frac{-1}{(\sqrt{2x+1})^3}$

$\therefore\ y=-\frac{1}{(\sqrt{2a+1})^3}(x-a)+\frac{1}{\sqrt{2a+1}}$

$x+(\sqrt{2a+1})^3y=3a+1$ 接線

原点からの距り

$d=\frac{3a+1}{\sqrt{1+(2a+1)^3}}$

(3) $V=\pi\int_0^a\left\{\frac{x}{a\sqrt{2a+1}}\right\}^2dx+\pi\int_a^{2a}\frac{1}{2x+1}dx-\pi\int_0^{2a}\left\{\frac{x}{2a\sqrt{4a+1}}\right\}^2dx$

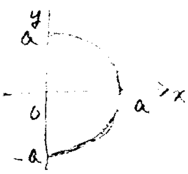
$=\pi\frac{1}{a^2(2a+1)}\left[\frac{x^3}{3}\right]_0^a+\frac{\pi}{2}\left[\log_e(2x+1)\right]_a^{2a}-\pi\frac{1}{4a^2(4a+1)}\left[\frac{x^3}{3}\right]_0^{2a}$

$=\pi\frac{a}{3(2a+1)}+\frac{\pi}{2}\log_e\frac{4a+1}{2a+1}-\pi\frac{2a}{3(4a+1)}$

$=\frac{\pi}{2}\log\frac{4a+1}{2a+1}+\pi\frac{a\{4a+1-4a-2\}}{3(2a+1)(4a+1)}$

$=\frac{\pi}{2}\log\frac{4a+1}{2a+1}-\frac{\pi a}{3(2a+1)(4a+1)}$

13.6



$x^2+y^2=a^2$ $y=\pm\sqrt{a^2-x^2}$

$\bar{x}=\frac{\int_0^a 2x\sqrt{a^2-x^2}\,dx}{\frac{1}{2}\pi a^2}=\frac{4\left[-\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}\right]_0^a}{\pi a^2}$

$=\frac{4a}{3\pi}$ $\bar{y}=0$

$\therefore$ 重心 $\left(\frac{4a}{3\pi a},\ 0\right)$

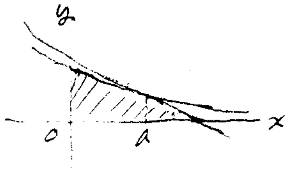
P55

13.7 $y = e^{-\frac{x}{k}}$ $(k>1$

(1) $y' = -\frac{1}{k}e^{\frac{x}{k}}$ $(a, e^{-\frac{a}{k}})$ における接線

$y - e^{-\frac{a}{k}} = -\frac{1}{k}e^{-\frac{a}{k}}(x-a)$

$y + \frac{1}{k}e^{-\frac{a}{k}}x = (\frac{a}{k}+1)e^{-\frac{a}{k}}$ $\quad ky + e^{-\frac{a}{k}}x = (a+k)e^{-\frac{a}{k}}$



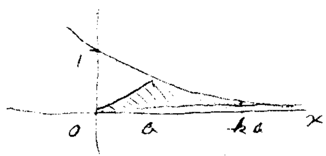
(2) $x=0,\ y=\frac{a+k}{k}e^{-\frac{a}{k}}$

$y=0 \quad x=a+k$

$(0, \frac{a+k}{k}e^{-\frac{a}{k}}) \quad (a+k, 0)$

$S = \frac{1}{2}(a+k)\frac{a+k}{k}e^{-\frac{a}{k}} = \frac{(a+k)^2}{2k}e^{-\frac{a}{k}}$

(3) $\lim_{a\to\infty} S = \lim_{a\to\infty}\frac{1}{2k}\frac{(a+k)^2}{e^{\frac{a}{k}}} = 0$



(4) $V = \pi\int_0^a \left(\frac{e^{-\frac{a}{k}}}{a}x\right)^2dx + \pi\int_a^{ka}\left(e^{-\frac{x}{k}}\right)^2dx$

$- \pi\int_0^{ka}\left(\frac{e^{-a}}{ka}x\right)^2dx$

$= \pi\frac{1}{a^2e^{\frac{2a}{k}}}\frac{a^3}{3} - \pi\frac{1}{k^2a^2e^{2a}}\frac{k^3a^3}{3} + \pi\left[-\frac{k}{2}e^{-\frac{2}{k}x}\right]_a^{ka}$

$= \frac{\pi a}{3}(e^{-\frac{2a}{k}} - ke^{-2a}) + \frac{\pi k}{2}(e^{-\frac{2a}{k}} - e^{-2a})$

13.8 (1) $y=f(x)$ $\quad (x_1, f(x_1))$ $\quad x_1>0 \quad f(x_1)>0$

接線 $y = f'(x_1)(x-x_1)+f(x_1)$ が $(ax_1, 0)$ $(a<0)$を通る

$0 = f'(x_1)(a-1)x_1 + f(x_1)$

∴ 微分方程式

$(a-1)xy' + y = 0$

(2) $\frac{y'}{y} = \frac{1}{(1-a)x}$ $\quad \log y = \frac{1}{1-a}\log|x| + C$

$y = Cx^{\frac{1}{1-a}}$