P. 56

第6章 偏微分法

§1. 偏微分

1.1 (1) $z=(x^2+y)^3$ $\quad z_x=6x(x^2+y)^2$ $\quad z_y=3(x^2+y)^2$

(2) $z=\log(\sin x+\cos y)$ $\quad z_x=\dfrac{\cos x}{\sin x+\cos y}$ $\quad z_y=\dfrac{-\sin y}{\sin x+\cos y}$

(3) $z=\sin^{-1}\dfrac{y}{\sqrt{x^2+y^2}}$ $\quad z_x=\dfrac{1}{\sqrt{1-\frac{y^2}{x^2+y^2}}}\ \dfrac{2xy}{-2(\sqrt{x^2+y^2})^3}=\dfrac{-y\,\mathrm{sign}(x)}{x^2+y^2}$

$\left(\mathrm{sign}(x)=\begin{cases}-1 & x<0\\ 1 & x>0\end{cases}\right)$ $\quad z_y=\dfrac{1}{\sqrt{1-\frac{y^2}{x^2+y^2}}}\left(\dfrac{1}{\sqrt{x^2+y^2}}-\dfrac{y^2}{(\sqrt{x^2+y^2})^3}\right)=\dfrac{|x|}{x^2+y^2}$

(4) $z=\dfrac{xy}{\sqrt{x^2+y^2}}$ $\quad z_x=\dfrac{y}{\sqrt{x^2+y^2}}-\dfrac{yx^2}{(\sqrt{x^2+y^2})^3}=\dfrac{y^3}{(\sqrt{x^2+y^2})^3}$

$z_y=\dfrac{x}{\sqrt{x^2+y^2}}-\dfrac{xy^2}{(\sqrt{x^2+y^2})^3}=\dfrac{y^3}{(\sqrt{x^2+y^2})^3}$

(5) $z=\dfrac{y}{x^2+y^2+1}$ $\quad z_x=\dfrac{-2xy}{(x^2+y^2+1)^2}$ $\quad z_y=\dfrac{x^2+y^2+1-2y^2}{(x^2+y^2+1)^2}=\dfrac{x^2-y^2+1}{(x^2+y^2+1)^2}$

(6) $z=\sin(x\cos y-y\sin x)$ $\quad z_x=\cos(x\cos y-y\sin x)(\cos y-y\cos x)$

$z_y=\cos(x\cos y-y\sin x)(-x\sin y-\sin x)$

(7) $z=\tan^{-1}\dfrac{y}{2x}$ $\quad z_x=\dfrac{1}{1+(\frac{y}{2x})^2}\ \dfrac{y}{-2x^2}=\dfrac{-2y}{4x^2+y^2}$

$z_y=\dfrac{1}{1+(\frac{y}{2x})^2}\ \dfrac{1}{2x}=\dfrac{2x}{4x^2+y^2}$

1.2 (1) $z=\tan^{-1}\dfrac{y}{x}$ $\quad z_x=\dfrac{1}{1+(\frac{y}{x})^2}\left(-\dfrac{y}{x^2}\right)=\dfrac{-y}{x^2+y^2}$ $\quad z_{xx}=\dfrac{2xy}{(x^2+y^2)^2}$

$z_y=\dfrac{1}{1+(\frac{y}{x})^2}\ \dfrac{1}{x}=\dfrac{x}{x^2+y^2}$ $\quad z_{yy}=\dfrac{-2xy}{(x^2+y^2)^2}$

$\therefore z_{xx}+z_{yy}=0$

(2) $z=\log(x^2+y^2)$ $\quad z_x=\dfrac{2x}{x^2+y^2}$ $\quad z_{xx}=\dfrac{2(y^2-x^2)}{(x^2+y^2)^2}$

$z_y=\dfrac{2y}{x^2+y^2}$ $\quad z_{yy}=\dfrac{2(x^2-y^2)}{(x^2+y^2)^2}$

$\therefore z_{xx}+z_{yy}=0$

(3) $z=\log\sqrt{x^2+y^2}=\frac{1}{2}\log(x^2+y^2)$ $\quad z_x=\dfrac{x}{x^2+y^2}$ $\quad z_{xx}=\dfrac{y^2-x^2}{(x^2+y^2)^2}$

$z_y=\dfrac{y}{x^2+y^2}$ $\quad z_{yy}=\dfrac{x^2-y^2}{(x^2+y^2)^2}$

$\therefore z_{xx}+z_{yy}=0$

P. 56

1.2 (5) $z = x^3y + 2x^2y^2 + 5xy^3$, $z_x = 3x^2y + 4xy^2 + 5y^3$ $z_{xx} = 6xy + 4y^2$

$z_y = x^3 + 4x^2y + 15xy^2$, $z_{yy} = 4x^2 + 30xy$

$\therefore z_{xx} + z_{yy} = 4(x^2 + y^2 + 9xy)$

(6) $z = \tan^{-1}\frac{x}{y}$ $z_x = \frac{y}{x^2+y^2}$ $z_{xx} = \frac{-2xy}{(x^2+y^2)^2}$

$z_y = \frac{-x}{x^2+y^2}$ $z_{yy} = \frac{2xy}{(x^2+y^2)^2}$

$\therefore z_{xx} + z_{yy} = 0$

(7) $z = e^{ax}(\sin by + \cos by)$ $z_{xx} = a^2e^{ax}(\sin by + \cos by)$

$z_{yy} = -b^2e^{ax}(\sin by + \cos by)$

$z_{xx} + z_{yy} = (a^2 - b^2)z$

1.3 $f = \frac{1}{\sqrt{x^2+y^2+z^2}}$ $f_x = \frac{-x}{(\sqrt{x^2+y^2+z^2})^3}$ $f_{xx} = \frac{3x^2 - (x^2+y^2+z^2)}{(\sqrt{x^2+y^2+z^2})^5}$

$f_y = \frac{-y}{(\sqrt{x^2+y^2+z^2})^3}$ $f_{yy} = \frac{3y^2 - (x^2+y^2+z^2)}{(\sqrt{x^2+y^2+z^2})^5}$

$f_z = \frac{-z}{(\sqrt{x^2+y^2+z^2})^3}$ $f_{zz} = \frac{3z^2 - (x^2+y^2+z^2)}{(\sqrt{x^2+y^2+z^2})^5}$

$\therefore f_{xx} + f_{yy} + f_{zz} = 0$

1.4 $z = e^xf(x+y) + e^{-x}g(x-y)$ $\frac{d}{dt}f(t) = f'(t)$, $\frac{d}{dt}g(t) = g'(t)$ とおく

$z_x = e^x\{f(x+y) + f'(x+y)\} + e^{-x}\{-g(x-y) + g'(x-y)\}$

$z_{xx} = e^x\{f(x+y) + 2f'(x+y) + f''(x+y)\} + e^{-x}\{g(x-y) - 2g'(x-y) + g''(x-y)\}$

$z_{yy} = e^xf''(x+y) + e^{-x}g''(x-y)$

$\therefore z_{xx} = z_{yy} + 2\{e^xf'(x+y) - e^{-x}g'(x-y)\} + e^xf(x+y) + e^{-x}g(x-y)$

$= z_{yy} + 2z_y + z$

1.5 $z = f(ax+by)$ $z_x = af'(ax+by)$ $z_y = bf'(ax+by)$

$\therefore bz_x - az_y = 0$

1.6 $z = f(x,y)$ $z_{xx} + z_{yy} = 0$ のとき $u = yz_x - xz_y$

$u_x = yz_{xx} - z_y - xz_{yx}$ $u_{xx} = yz_{xxx} - 2z_{xy} - xz_{yxx}$

$u_y = z_x + yz_{xy} - xz_{yy}$ $z_{yy} = 2z_{xy} + yz_{xyy} - xz_{yyy}$

P.56

$$\therefore\ u_{xx}+u_{yy} = y(z_{xxx}+z_{xyy}) - x(z_{yxx}+z_{yyy})$$
$$= y\frac{\partial}{\partial x}(z_{xx}+z_{yy}) - x\frac{\partial}{\partial y}(z_{xx}+z_{yy}) = 0$$

1.7 $z=\cos(xy)$

$$z_x = -\sin(xy)\,y \qquad z_{xy} = -\cos(xy)xy - \sin(xy)$$
$$= -(xy\cos(xy)+\sin(xy))$$

1.8
$$u=\begin{vmatrix}1&1&1\\x&y&z\\x^2&y^2&z^2\end{vmatrix} \qquad u_x=\begin{vmatrix}0&1&1\\1&y&z\\2x&y^2&z^2\end{vmatrix} \qquad u_{xx}=\begin{vmatrix}0&1&1\\0&y&z\\2&y^2&z^2\end{vmatrix}$$

同様に
$$u_{yy}=\begin{vmatrix}1&0&1\\x&0&z\\x^2&2&z^2\end{vmatrix} \qquad u_{zz}=\begin{vmatrix}1&1&0\\x&y&0\\x^2&y^2&2\end{vmatrix}$$

$$\therefore\ u_{xx}+u_{yy}+u_{zz} = \begin{vmatrix}1&1&1\\x&y&z\\2&2&2\end{vmatrix} = 0$$

1.9 $x+y+z=f(x^2+y^2+z^2)$ $\qquad \frac{d}{d\alpha}f(\alpha)=f'(\alpha)$ とおく

(1) $1+z_x = 2xf'(x^2+y^2+z^2) + 2zz_xf'(x^2+y^2+z^2)$

$$z_x\{1-2zf'(x^2+y^2+z^2)\} = 2xf'(x^2+y^2+z^2)-1$$

$$z_x = \frac{2xf'(x^2+y^2+z^2)-1}{1-2zf'(x^2+y^2+z^2)}$$

(2) $z_y = \dfrac{2yf'(x^2+y^2+z^2)-1}{1-2zf'(x^2+y^2+z^2)}$

$$(y-z)z_x+(z-x)z_y = \frac{2x(y-z)f'(x^2+y^2+z^2)-(y-z)+2y(z-x)f'(x^2+y^2+z^2)-(z-x)}{1-2zf'(x^2+y^2+z^2)}$$

$$= \frac{2z(y-x)f'(x^2+y^2+z^2)-(y-x)}{1-2zf'(x^2+y^2+z^2)} = -(y-x)$$

$$\therefore\ (y-z)z_x+(z-x)z_y = x-y$$

1.10 $z=f(r)$ $\quad r=\sqrt{x^2+y^2}$

$$z_x = f'(r)\frac{x}{\sqrt{x^2+y^2}} \qquad z_{xx} = f''(r)\frac{x^2}{x^2+y^2} + f'(r)\left\{\frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(\sqrt{x^2+y^2})^3}\right\}$$
$$= f''(r)\frac{x^2}{r^2}+f'(r)\frac{y^2}{r^3}$$
$$z_{xy} = f''(r)\frac{xy}{r^2} - f'(r)\frac{xy}{(\sqrt{x^2+y^2})^3}$$
$$= f''(r)\frac{xy}{r^2} - f'(r)\frac{xy}{r^3}$$
$$z_{yy} = f''(r)\frac{y^2}{r^2} + f'(r)\frac{x^2}{r^3}$$

P.56

1.11 1.9に同じ

1.12 $f(x,y)=x^3 a^{\sin y}$ $f_x=3x^2 a^{\sin y}$ $f_y=x^3 \log a\, e^{\sin y}\cos y$

$=x^3 a^{\sin y}(\log a\cdot\cos y)$

$f_{xy}=3x^2 \log a\, a^{\sin y}\cos y$

$=3x^2 a^{\sin y}(\log a\cdot\cos y)$

1.13 $u=\log(x^3+y^3+z^3-3xyz)$

(1) $u_x=\dfrac{3(x^2-yz)}{x^3+y^3+z^3-3xyz}$

(2) $u_x+u_y+u_z=\dfrac{3(x^2-yz+y^2-xz+z^2-xy)}{x^3+y^3+z^3-3xyz}$

$=\dfrac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y)^3+z^3-3xy(x+y+z)}$

$=\dfrac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)\{(x+y)^2-(x+y)z+z^2-3xy\}}$

$=\dfrac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-zx-yz)}$

$=\dfrac{3}{x+y+z}$

P.57

§2 合成関数の偏微分

2.1 (1) $z=\tan^{-1}\frac{y}{x}$, $x=t+\sin t$, $y=1-\cos t$

$$\frac{dz}{dt}=\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)\cdot(1+\cos t)+\frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x}\sin t$$
$$=\frac{-y(1+\cos t)+x\sin t}{x^2+y^2}$$

(2) $z=f(x,y)$ $x=g(t)$ $y=h(t)$

$$\frac{dz}{dt}=f_x(x,y)g'(t)+f_y(x,y)h'(t)$$

2.2 (1) $z=x^2+y^2$, $x=2u-v$ $y=u+2v$

$$z_u=2x\cdot 2+2y\cdot 1=4x+2y=10u$$
$$z_v=2x\cdot(-1)+2y\cdot 2=2(-x+2y)=10v$$

(2) $z=xy$, $x=\log\sqrt{u^2+v^2}=\frac{1}{2}\log(u^2+v^2)$ $y=\tan^{-1}\frac{u}{v}$

$$z_u=yx_u+xy_u=y\frac{u}{u^2+v^2}+x\frac{1}{1+\frac{u^2}{v^2}}\frac{1}{v}$$
$$=\frac{yu+xv}{u^2+v^2}=\frac{1}{u^2+v^2}\left(u\tan^{-1}\frac{u}{v}+v\log\sqrt{u^2+v^2}\right)$$
$$z_v=yx_v+xy_v=y\frac{v}{u^2+v^2}+x\frac{-u}{u^2+v^2}$$
$$=\frac{1}{u^2+v^2}\left(v\tan^{-1}\frac{u}{v}-u\log\sqrt{u^2+v^2}\right)$$

2.3 $z=f(x,y)$ $x=u+v$ $y=uv$

$$z_u=z_x+z_y v$$
$$z_{uv}=z_{xx}+z_{xy}u+z_{yx}v+z_{yy}uv+z_y$$
$$=z_{xx}+(u+v)z_{xy}+z_{yy}uv+z_y$$
$$=z_{xx}+xz_{xy}+yz_{yy}+z_y$$

2.4 $z=f(x,y)$ $x=r\cos\theta$ $y=r\sin\theta$

$$z_r=z_x\cos\theta+z_y\sin\theta$$
$$z_{rr}=z_{xx}\cos^2\theta+2z_{xy}\cos\theta\sin\theta+z_{yy}\sin^2\theta$$
$$z_\theta=z_x(-r\sin\theta)+z_y r\cos\theta$$
$$z_{\theta\theta}=z_{xx}r^2\sin^2\theta-2z_{xy}r^2\sin\theta\cos\theta+z_{yy}r^2\cos^2\theta-z_x r\cos\theta-z_y r\sin\theta$$
$$\therefore\ z_{rr}+\frac{1}{r^2}z_{\theta\theta}=z_{xx}+z_{yy}-\frac{1}{r}z_r$$
$$\therefore\ z_{xx}+z_{yy}=z_{rr}+\frac{1}{r^2}z_{\theta\theta}+\frac{1}{r}z_r$$

P.57

2.5 $z=f(x,y) \quad x=r\cos\theta \quad y=r\sin\theta$

$z_r = z_x\cos\theta + z_y\sin\theta$

$z_\theta = -rz_x\sin\theta + rz_y\cos\theta$

$\therefore (z_r)^2 + \frac{1}{r^2}(z_\theta)^2 = (z_x\cos\theta + z_y\sin\theta)^2 + (-z_x\sin\theta + z_y\cos\theta)^2$

$= (z_x)^2 + (z_y)^2$

2.6 $u=2x+3y, \quad v=4x-5y, \quad z=f(u,v)$

$5u+3v=22x$ $\qquad \begin{cases} x = \frac{1}{22}(5u+3v) \\ y = \frac{1}{11}(2u-v) \end{cases}$

$2u-v=11y$

$z_u = z_x\frac{5}{22} + z_y\frac{2}{11} = \frac{1}{22}(5z_x+4z_y)$

$z_v = z_x\frac{3}{22} + z_y\frac{-1}{11} = \frac{1}{22}(3z_x-2z_y)$

2.7 $z=f(x,y) \quad x=\phi\cos\alpha - \rho\sin\alpha, \ y=\phi\sin\alpha+\rho\cos\alpha$ (問題ミスプリント)

$z_\phi = z_x\cos\alpha + z_y\sin\alpha \qquad z_{\phi\phi} = z_{xx}\cos^2\alpha + 2z_{xy}\cos\alpha\sin\alpha + z_{yy}\sin^2\alpha$

$z_\rho = z_x(-\sin\alpha) + z_y\cos\alpha \qquad z_{\rho\rho} = z_{xx}\sin^2\alpha - 2z_{xy}\cos\alpha\sin\alpha + z_{yy}\cos^2\alpha$

$\therefore z_{\phi\phi} + z_{\rho\rho} = z_{xx} + z_{yy}$

2.8 $z=f(r) \qquad r=\sqrt{x^2+2y^2}$

$z_x = f'(r)\frac{x}{\sqrt{x^2+2y^2}} \qquad z_{xx} = f''(r)\frac{x^2}{x^2+2y^2} + f'(r)\frac{1}{\sqrt{x^2+2y^2}} - f'(r)\frac{x^2}{(\sqrt{x^2+2y^2})^3}$

$= f''(r)\frac{x^2}{r^2} + f'(r)\frac{2y^2}{r^3}$

$z_y = f'(r)\frac{2y}{\sqrt{x^2+2y^2}} \qquad z_{yy} = f''(r)\frac{4y^2}{r^2} + f'(r)\frac{2}{\sqrt{x^2+2y^2}} - f'(r)\frac{4y^2}{(\sqrt{x^2+2y^2})^3}$

$= f''(r)\frac{4y^2}{r^2} + f'(r)\frac{2x^2}{r^3}$

$\therefore z_{xx} + \frac{1}{2}z_{yy} = f''(r)\frac{x^2+2y^2}{r^2} + f'(r)\frac{2y^2+x^2}{r^3} = f''(r) + f'(r)\frac{1}{r}$

$\therefore z_{xx} + \frac{1}{2}z_{yy} = f''(r) + \frac{f'(r)}{r}$

P 57

§3 連続と偏微分

3.1 $$f(x,y)=\begin{cases} x+y+\dfrac{xy}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

(1) $f_x(0,0)=\lim_{h\to 0}\dfrac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\dfrac{h-0}{h}=1$

(2) $y=x$ 上で $x\to 0$ のとき

$$\lim_{y=x\to 0} f(x,y)=\lim_{x\to 0}\left(2x+\frac{x^2}{2x^2}\right)=\frac{1}{2}\neq f(0,0)$$

∴ 原点で不連続

3.2 $$f(x,y)=\frac{\log|ax^2+by^2-1|}{x^2+y^2}$$

(1) $$\lim_{y\to 0}\left\{\lim_{x\to 0}\frac{\log|ax^2+by^2-1|}{x^2+y^2}\right\}=\lim_{y\to 0}\frac{\log(by^2-1)}{y^2}$$

$$=\lim_{y\to 0}\frac{\frac{2by}{by^2-1}}{2y}=\lim_{y\to 0}\frac{b}{by^2-1}=-b$$

(2) (1) と同様に

$$\lim_{x\to 0}\left\{\lim_{y\to 0}\frac{\log|ax^2+by^2-1|}{x^2+y^2}\right\}=-a$$

∴ $a=b$

3.3 $$\Delta u=\lim_{h\to 0}\{u(x-h,y)+u(x,y+h)+u(x+h,y)+u(x,y-h)-4u(x,y)\}/h^2$$

$$=\lim_{h\to 0}\frac{1}{h^2}\{u(x-h,y)-u(x,y)+u(x,y+h)-u(x,y)+u(x+h,y)-u(x,y)+u(x,y-h)-u(x,y)\}$$

$$=\lim_{h\to 0}\frac{1}{h^2}\{-hu_x(x,y)+\frac{h^2}{2}u_{xx}(x-\theta_1 h,y)+hu_y(x,y)+\frac{h^2}{2}u_{yy}(x,y+\theta_2 h)$$
$$+hu_x(x,y)+\frac{h^2}{2}u_{xx}(x+\theta_3 x,y)-hu_y(x,y)+\frac{h^2}{2}u_{yy}(x,y-\theta_4 h)\}$$

$$=u_{xx}+u_{yy}$$

3.4 $$f(x,y)=\begin{cases} \dfrac{xy}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

$\lim_{x=y\to 0} f(x,y)=\lim_{x\to 0}\dfrac{x^2}{2x^2}=\dfrac{1}{2}\neq f(0,0)$ ∴ $(0,0)$ で不連続

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=0$$

$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=0$$

P57

3.5

$$f(x,y)=\begin{cases}\dfrac{xy(e^{x^2}-e^{y^2})}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

(1)
$$f_x(x,y)=\frac{y(e^{x^2}-e^{y^2})(x^2+y^2)+2x^2y(x^2+y^2)e^{x^2}-2x^2y(e^{x^2}-e^{y^2})}{(x^2+y^2)^2}$$
$$=\frac{y(e^{x^2}-e^{y^2})(y^2-x^2)+2x^2y(x^2+y^2)e^{x^2}}{(x^2+y^2)^2}$$

(2)
$$f_y(x,y)=\frac{\{x(e^{x^2}-e^{y^2})-2xy^2e^{y^2}\}(x^2+y^2)-2xy^2(e^{x^2}-e^{y^2})}{(x^2+y^2)^2}$$
$$=\frac{x(x^2-y^2)(e^{x^2}-e^{y^2})-2xy^2(e^{x^2}-e^{y^2})}{(x^2+y^2)^2}$$

(3)
$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=0$$
$$f_y(0,0)=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}=0$$
$$f_{xy}(0,0)=\lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k}=\lim_{k\to 0}\frac{k(1-e^{k^2})k^2}{k^5}=-1 \quad *$$
$$f_{yx}(0,0)=\lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h}=\lim_{h\to 0}\frac{h^3(e^{h^2}-1)}{h^5}$$
$$=\lim_{h\to 0}\frac{1+h^2+o(h^4)-1}{h^2}=1$$

$$* \quad e^{k^2}=1+k^2+\frac{k^4}{2!}+\frac{k^6}{3!}+\cdots$$
$$=1+k^2+o(k^4)$$

$o(k^4)$ 4位の無限小

P.58 # §4 偏微分の応用1(最大・最小)

4.1(1) $z=x^4-x^3-y^3+3x^2y$

$z_x=4x^3-3x^2+6xy=x(4x^2-3x+6y)$

$z_y=-3y^2+3x^2=3(x-y)(x+y)$

$z_{xx}=12x^2-6x+6y$

$z_{xy}=6x$

$z_{yy}=-6y$

$x(4x^2-3x+6y)=0$

$y=\pm x$

$y=x$ のとき　$x^2(4x+3)=0$

$y=-x$　$x^2(4x-9)=0$

$\therefore (0,0),\ (-\frac{3}{4},-\frac{3}{4})\ (\frac{9}{4},-\frac{9}{4})$

$\frac{1}{36}(z_{xx}z_{yy}-z_{xy}^2)=-y(2x^2-x+y)-x^2$

$(0,0)$ のとき　0

$(-\frac{3}{4},-\frac{3}{4})$ のとき　$\frac{3}{4}\cdot\frac{9}{8}-\frac{9}{16}>0$　$f_{xx}>0$

$\therefore (-\frac{3}{4},-\frac{3}{4})$ のとき 極小値　$\frac{81}{4^4}+\frac{27}{4^3}+\frac{27}{4^3}-\frac{81}{4^3}=\frac{-27}{4^4}$

$(\frac{9}{4},-\frac{9}{4})$ のとき　$\frac{9}{4}(\frac{81}{8}-\frac{9}{4}-\frac{9}{4})-\frac{81}{16}=\frac{405-162}{32}>0$　$f_{xx}>0$

$(\frac{9}{4},-\frac{9}{4})$ のとき 極小値　$(\frac{9}{4})^4-(\frac{9}{4})^3+(\frac{9}{4})^3-3(\frac{9}{4})^3=\frac{-3^7}{4^4}$

(2) $z=x^3-2x^2y+x^2-y^2$

$z_x=3x^2-4xy+2x=x(3x-6y+2)$

$z_y=-2x^2-2y=-2(y+x^2)$

$x(3x-4y+2)=0$

$y+x^2=0$

$x=0$　$3x+4x^2+2=0$ 解なし

$(0,0)$

$z_{xx}=6x-6y+2$

$z_{xy}=-4x$

$z_{yy}=-2$

$z_{xx}z_{yy}-z_{xy}^2=-2(6x-6y+2)-16x^2$

$(0,0)$ で 負

$\therefore$ 極値なし

(3) $z=x^3+2x^2y-xy^2-4xy$

$z_x=3x^2+4xy-y^2-4y$

$z_y=2x^2-2xy-4x=2x(x-y-2)$

$\begin{cases} 3x^2+4xy-y^2-4y=0 \\ x(x-y-2)=0 \end{cases}$

$x=0$　$y=0$　$y=-4$

$y=x-2$

$3x^2+4x^2-8x-x^2+4x-4-4x+8$

$6x^2-8x+4=0$

解なし

$\therefore (0,0),\ (0,-4)$

$z_{xx}=6x+4y$

$z_{xy}=4x-2y-4$

$z_{yy}=-2x$

$z_{xx}z_{yy}-z_{xy}^2=-4(3x+2y)x-4(2x-y-1)^2$

$(0,0)$ のとき <0，$(0,-4)$ のとき <0

$\therefore$ 極値なし

P.58

4.1 (4) $z = x^4 + y^4 - 2x^2 - 2y^2 + 4xy$

$z_x = 4x^3 - 4x - 4y$

$z_y = 4y^3 - 4y - 4x$

$x^3 - x - y = 0$
$y^3 - y - x = 0$
$x^3 - y^3 = 0 \quad (x-y)(x^2+xy+y^2)=0$
$y = x$, $(0,0)$
$x^3 - 2x = 0$, $x = 0, \pm\sqrt{2}$

$(0,0)$ $(\sqrt{2}, \sqrt{2})$ $(-\sqrt{2}, -\sqrt{2})$

$z_{xx} = 12x^2 - 4$

$z_{xy} = -4$

$z_{yy} = 12y^2 - 4$

$D = z_{xx} z_{yy} - z_{xy}^2 = 16(3x^2-1)(3y^2-1) - 16$

$(0,0)$のとき $D = 0$ $(\pm\sqrt{2}, \pm\sqrt{2})$のとき $D > 0$

$(0,0)$に近い点で $z > 0$, $z < 0$ となり得るので極値をとらない

$(\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$ で $f_{xx} > 0$ ∴ 極小値 $4+4-4\cdot 2-8 = -8$ をとる.

(5) $z = (x^2+y^2)^2 - 2(x^2-y^2)$

$z_x = 4x(x^2+y^2) - 4x = 4x(x^2+y^2-1) = 0$

$z_y = 4y(x^2+y^2) + 4y = 4y(x^2+y^2+1) = 0$

$(0,0)$ $(1,0)$ $(-1,0)$

$z_{xx} = 12x^2 + 4y^2 - 4$

$z_{xy} = 8xy$

$z_{yy} = 12y^2 + 4x^2 + 4$

$D = 16(3x^2+y^2-1)(3y^2+x^2+1) - 16\cdot 4xy$
$= 16\{(3x^2+y^2-1)(3y^2+x^2+1) - 4xy\}$

$(0,0)$のとき $D < 0$ $f_{xx} < 0$ ∴ 極値なし

$(1,0)$のとき $D > 0$ $f_{xx} > 0$ ∴ 極小値 -1

$(-1,0)$ 〃 $D > 0$ $f_{xx} > 0$ 極小値 -1

(6) $z = x^3 + 3xy + y^3$

$z_x = 3x^2 + 3y$

$z_y = 3x + 3y^2$

$x^2 + y = 0$
$x + y^2 = 0$

$y = -x^2$ $x(1+x^3) = 0$

$(0,0)$ $(-1,-1)$

$z_{xx} = 6x$

$z_{yy} = 6y$

$z_{xy} = 3$

$D = 9(4xy - 1)$

$(0,0)$で $D < 0$ 極値なし $(-1,-1)$で $D > 0$ $z_{xx} < 0$

∴ $(-1,-1)$で極大値 1

(7) $z = xy - x^3y - xy^3$

$z_x = y - 3x^2y - y^3 = y(1 - 3x^2 - y^2)$

$z_y = x - x^3 - 3xy^2 = x(1 - 3y^2 - x^2)$

$y(1-3x^2-y^2) = 0$
$x(1-3y^2-x^2) = 0$

$(0,0)$ $(\pm\frac{1}{2}, \pm\frac{1}{2})$ $(\pm 1, 0)$ $(0, \pm 1)$

p.58

$z_{xx} = -6xy$

$z_{xy} = 1-3x^2-3y^2$ $\qquad D = 36x^2y^2-(1-3x^2-3y^2)^2$

$z_{yy} = -6xy$

$(0,0)$ のとき $D<0$ 極値なし

$(\pm 1, 0)$ のとき $D<0$ 〃

$(0, \pm 1)$ 〃 〃 〃

$(\pm\frac{1}{2}, \pm\frac{1}{2})$ のとき $D = \frac{9}{4} - (1-\frac{3}{2})^2 > 0$

$(\frac{1}{2}, \frac{1}{2})$ $(-\frac{1}{2}, -\frac{1}{2})$ のとき $z_{xx} < 0$ 極大値 $\frac{1}{8}$

$(\frac{1}{2}, -\frac{1}{2})$ $(-\frac{1}{2}, \frac{1}{2})$ $z_{xx} > 0$ 極小値 $-\frac{1}{8}$

(8) $z = x^3+8y^3+12axy \quad (a \neq 0)$

$z_x = 3x^2+12ay = 3(x^2+4ay)$

$z_y = 24y^2+12ax = 12(2y^2+ax)$

$x^2+4ay=0 \quad y = -\frac{x^2}{4a}$

$2y^2+ax=0 \quad \frac{x^4}{8a^2}+ax=0$

$x(x^3+8a^3)=0$

$(0,0)$ $(-2a, -a)$

$\begin{cases} z_{xx} = 6x \\ z_{yy} = 48y \\ z_{xy} = 12a \end{cases}$ $\qquad D = 12^2(2xy-a^2)$

$(0,0)$ のとき $D<0$ 極値なし

$(-2a, -a)$ のとき $D>0$

$a>0$ のとき $z_{xx} = -6a < 0$ 極大値 $8a^3$

$a<0$ のとき $z_{xx} = -6a > 0$ 極小値 $8a^3$

(9) $z = xy(x+2y-6) = x^2y+2xy^2-6xy$

$z_x = 2xy+2y^2-6y = (2x+2y-6)y$

$z_y = x^2+4xy-6x = (x+4y-6)x$

$(x+y-3)y=0$

$(x+4y-6)x=0$

$x=0 \quad y=0 \quad y=1$

$y=0, 3 \quad x=6 \quad x=2$

$(0,0), (0,3), (6,0)$ $(2,1)$

$z_{xx} = 2y \quad z_{xy} = 2x+4y-6 \quad z_{yy} = 4x$

$D = z_{xx}z_{yy} - z_{xy}^2 = 8xy - 4(x+2y-3)^2 = 4\{2xy-(x+2y-3)^2\}$

$(0,0)$ で $D<0$, $(0,3)$ で $D<0$ $(6,0)$ で $D<0$ $(2,1)$ で $D>0$

$(2,1)$ で $z_{xx} > 0$ ∴ $(2,1)$ で極小値 -4

P.58

(10) $z = x^3 + 2x^2y + xy^2 - 4xy$

$z_x = 3x^2 + 4xy + y^2 - 4y$

$z_y = 2x^2 + 2xy - 4x$

$3x^2 + 4xy + y^2 - 4y = 0$

$x^2 + xy - 2x = 0$

$x(x+y-2) = 0$

$x = 0$ $\quad y = 2-x$

$y = 0, 4$ $\quad 3x^2 + 4x(2-x) + (2-x)^2 - 4(2-x) = 0$

$8x - 4 = 0$

$x = \frac{1}{2}$ $\quad y = \frac{3}{2}$

$(0,0)$ $\quad (0,4)$ $\quad (\frac{1}{2}, \frac{3}{2})$

$z_{xx} = 6x + 4y$

$z_{xy} = 4x + 2y - 4$

$z_{yy} = 2x$

$D = 4(3x+2y)x - 4(2x+y-2)^2$

$= 4\{(3x+2y)x - (2x+y-2)^2\}$

$(0,0)$ で $D<0$ $\quad (0,4)$ で $D<0$ $\quad (\frac{1}{2}, \frac{3}{2})$ で $\frac{D}{4} = \frac{9}{4} - \frac{1}{4} > 0$

$z_{xx} = \frac{6}{2} + \frac{12}{2} > 0$

$\therefore (\frac{1}{2}, \frac{3}{2})$ で 極小値 -1.

(11) $z = 4x^3 - y^3 + 3x^2y + 9y$

$z_x = 12x^2 + 6xy$

$z_y = -3y^2 + 3x^2 + 9$

$6x(2x+y) = 0$ $\quad x = 0$ $\quad y = -2x$

$-y^2 + x^2 + 3 = 0$ $\quad -3x^2 + 3 = 0$

$(0, \pm\sqrt{3})$ $\quad (1, -2), (-1, 2)$

$z_{xx} = 24x + 6y$

$z_{xy} = 6x$

$z_{yy} = -6y$

$D = 36\{-y(4x+y) - x^2\}$

$(0, \pm\sqrt{3})$ $D<0$ $\quad (1, -2)$ $D>0$ $\quad (-1, 2)$, $D>0$

$\therefore (1, -2)$ $\quad z_{xx} = 24 - 12 > 0$ $\quad (-1, 2)$ $\quad z_{xx} = -24 + 12 < 0$

$(1, -2)$ で 極小値 $\quad 4 + 8 - 6 - 18 = -12$

$(-1, 2)$ で 極大値 $\quad -4 - 8 + 6 + 18 = 12$

(12) $z = x^2 - xy + y^2 - 3x + y - 2$

$z_x = 2x - y - 3$

$z_y = -x + 2y + 1$

$2x - y - 3 = 0$

$-2x + 4y + 2 = 0$

$3y - 1 = 0$

$y = \frac{1}{3}$ $\quad x = \frac{5}{3}$

$z_{xx} = 2$ $\quad z_{xy} = -1$ $\quad z_{yy} = 2$

$D = z_{xx} z_{yy} - z_{xy}^2 = 4 - 1 > 0$ $\quad z_{xx} > 0$

$\therefore (\frac{5}{3}, \frac{1}{3})$ で 極小値 $\quad \frac{25}{9} - \frac{5}{9} + \frac{1}{9} - 5 + \frac{1}{3} - 2 = -\frac{13}{3}$

P58

(13) $z=x^4+y^4-3(x-y)^2$

$z_x=4x^3-6(x-y)$

$z_y=4y^3+6(x-y)$

$2x^3-3(x-y)=0$

$2y^3+3(x-y)=0$

$x^3+y^3=0$

$y=-x$

$2(x^3-y^3)-6(x-y)=0$

$(x-y)(x^2+xy+y^2-3)=0$

$y=x,\ x^2+xy+y^2=3$

$(0,0)\ (\sqrt{3},-\sqrt{3})\ (-\sqrt{3},\sqrt{3})$

$z_{xx}=12x^2-6$

$z_{xy}=6$

$z_{yy}=12y^2-6$

$\frac{D}{36}=(2x^2-1)(2y^2-1)-1$

$(0,0)$で $D=0$　$(0,0)$の近くで正にも負にもなるから極値なし

$(\pm\sqrt{3},\mp\sqrt{3})$で $D>0$　$z_{xx}>0$

$\therefore (\pm\sqrt{3},\mp\sqrt{3})$で極小値 $9+9-36=-18$

(14) $z=x(1-x^2-y^2)=x-x^3-xy^2$

$z_x=1-3x^2-y^2$

$z_y=-2xy$

$x=0$　$y=0$

$y=\pm 1$　$x=\pm\frac{1}{\sqrt{3}}$

$z_{xx}=-6x$

$z_{xy}=-2y$

$z_{yy}=-2x$

$D=12x^2-4y^2$　$\frac{D}{4}=3x^2-y^2$

$(0,0)$で $D=0$ 極値なし

$(0,\pm 1)$で $D<0$ 極値なし　$(\pm\frac{1}{\sqrt{3}},0)$で $D>0$　$z_{xx}=-6(\pm\frac{1}{\sqrt{3}})$

$(\frac{1}{\sqrt{3}},0)$で極大値 $\frac{2}{3\sqrt{3}}$　$(\frac{-1}{\sqrt{3}},0)$で極小値 $\frac{-2}{3\sqrt{3}}$

42 (1) $z=(x^2+2y^2)e^{-(x^2+y^2)}$

$z_x=\{2x-2x(x^2+2y^2)\}e^{-(x^2+y^2)}$

$z_y=\{4y-2y(x^2+2y^2)\}e^{-(x^2+y^2)}$

$x(1-x^2-2y^2)=0$

$y(2-x^2-2y^2)=0$

$x=0$　$y=0$

$y=\pm 1$　$x=\pm 1$

$\therefore (0,0),\ (0,\pm 1)\ (\pm 1,0)$

$z_{xx}=\{2-6x^2-4y^2-4x^2+4x^2(x^2+2y^2)\}e^{-(x^2+y^2)}$

$z_{yy}=\{4-2x^2-12y^2-8y^2+4y^2(x^2+y^2)\}e^{-(x^2+y^2)}$

$z_{xy}=\{-8xy-4xy+4xy(x^2+y^2)\}e^{-(x^2+y^2)}$

$D=z_{xx}z_{yy}-z_{xy}^2$ とおくと

$(0,0)$で $D>0$　$z_{xx}>0$　極小値 0

$(0,\pm 1)$で $D>0$　$z_{xx}<0$　極大値 $2e^{-1}$

$(\pm 1,0)$で $D<0$　極値なし

P.58

4.2 (2) $z = xe^{-(x^2+y^2)}$

$z_x = (1-2x^2)e^{-(x^2+y^2)}$ 　　 $(\pm\frac{1}{\sqrt{2}}, 0)$

$z_y = -2xy\,e^{-(x^2+y^2)}$

$z_{xx} = \{-4x - 2x(1-2x^2)\}e^{-(x^2+y^2)}$

$z_{xy} = -2y(1-2x^2)e^{-(x^2+y^2)}$

$z_{yy} = (-2x + 4xy^2)e^{-(x^2+y^2)}$ 　　 $D = z_{xx}z_{yy} - z_{xy}^2$ とおく

$(\pm\frac{1}{\sqrt{2}}, 0)$ のとき $D > 0$

$(\frac{1}{\sqrt{2}}, 0)$ のとき $z_{xx} < 0$ 　極大値 $\frac{1}{\sqrt{2}}e^{-\frac{1}{2}}$

$(-\frac{1}{\sqrt{2}}, 0)$ 〃 $z_{xx} > 0$ 　極小値 $-\frac{1}{\sqrt{2}}e^{-\frac{1}{2}}$

(3) $z = \sin x + \sin y + \sin(x+y)$ 　　 $\cos x - \cos y = 0$ 　 $2\sin\frac{x+y}{2}\sin\frac{x-y}{2} = 0$

$z_x = \cos x + \cos(x+y)$ 　　 $\frac{x+y}{2} = 0, \pi$ 　 $\frac{x-y}{2} = 0$

$z_y = \cos y + \cos(x+y)$, $0 < x, y < \pi$ 　　 $y = 2\pi - x$ 　 $y = x$

$y = -x$ 　 $\cos x + \cos 2x = 0$

$(\frac{\pi}{3}, \frac{\pi}{3})$ 　　 $\cos x + 1 = 0$ 　 $2\cos^2 x + \cos x - 1 = 0$

$z_{xx} = -\sin x - \sin(x+y)$ 　　 $x = \pi$ 　 $\cos x = \frac{1}{2}, -1$

$z_{yy} = -\sin y - \sin(x+y)$

$z_{xy} = -\sin(x+y)$ 　　 $D = z_{xx}z_{yy} - z_{xy}^2 > 0$

$\therefore (\frac{\pi}{3}, \frac{\pi}{3})$ で $z_{xx} < 0$ 　極大値 $\frac{3\sqrt{3}}{2}$

4.3 $f(x, y) = x^3 - 3xy + y^3$

(1) $f_x = 3x^2 - 3y$ 　 $f_y = 3y^2 - 3x$ 　 $f_{xx} = 6x$ 　 $f_{xy} = -3$ 　 $f_{yy} = 6y$

$(f_{xxx} = 6 \quad f_{xxy} = f_{yyx} = 0 \quad f_{yyy} = 6)$

$\therefore F(x, y) = f(x+h, y+k) - f(x, y)$

$= 3(x^2-y)h + 3(y^2-x)k + \frac{1}{2}(6xh^2 - 6hk + 6yk^2)$

$= 3\{(x^2-y)h + (y^2-x)k + xh^2 - hk + yk^2\}$

(2) (1) より $x^2 - y = 0$ 　 $y^2 - x = 0$ 　 $y(y^3-1) = 0$ 　 $y = 0, 1$
$x = 0, 1$

$(0, 0)$ 　 $(1, 1)$

$F(0,0) = -3hk$, $F(1,1) = 3(h^2 - hk + k^2) > 0$

$\therefore (1,1)$ で極小値 -1.

P58

4.4 $f(x,y) = \begin{vmatrix} \sin x & 0 & \sin y \\ 1 & 1 & -\cos y \\ -\cos x & 1 & 1 \end{vmatrix} = \sin x + \sin y + \sin y\cos x + \cos y\sin x$

$f_x(x,y) = \begin{vmatrix} \cos x & 0 & \sin y \\ 0 & 1 & -\cos y \\ \sin x & 1 & 1 \end{vmatrix} = \cos x - \sin x\sin y + \cos x\cos y$

$f_y(x,y) = \begin{vmatrix} \sin x & 0 & \cos y \\ 1 & 1 & \sin y \\ -\cos x & 1 & 0 \end{vmatrix} = \cos y + \cos x\cos y - \sin x\sin y$

$\cos x + \cos(x+y) = 0$ $\cos x - \cos y = 0$

$\cos y + \cos(x+y) = 0$ $\sin\frac{x+y}{2}\sin\frac{y-x}{2} = 0$

∴ $x+y = 2n\pi$ $y-x = 2n\pi$

$y = 2n\pi - x$ のとき $\cos x + \cos(2n\pi) = 0$ $x = (2m-1)\pi$ $y = (2k-1)\pi$

$y = 2n\pi + x$ のとき $\cos x + \cos 2x = 0$ $2\cos^2 x + \cos x - 1 = 0$

$(2\cos x - 1)(\cos x + 1) = 0$ $x = \pm\frac{\pi}{3}$, $x = (2k-1)\pi$

∴ $((2m-1)\pi, (2k-1)\pi)$, $((2n\pm\frac{1}{3})\pi, (2m\pm\frac{1}{3})\pi)$

$f_{xx} = -\sin x - \sin(x+y)$

$f_{xy} = -\sin(x+y)$ $D = f_{xx}f_{yy} - f_{xy}^2$

$f_{yy} = -\sin y - \sin(x+y)$ $= \{\sin x + \sin(x+y)\}\{\sin y + \sin(x+y)\} - \sin^2(x+y)$

$= \sin x\sin y + (\sin x + \sin y)\sin(x+y)$

∴ $((2n-1)\pi, (2k-1)\pi)$ のとき 極値なし

$((2n+\frac{1}{3})\pi, (2k+\frac{1}{3})\pi)$ のとき 極大値 $\frac{3\sqrt{3}}{2}$

$((2n-\frac{1}{3})\pi, (2k-\frac{1}{3})\pi)$ のとき 極小値 $-\frac{3\sqrt{3}}{2}$

4.5 $x^2 + xy + y^2 = 3$

(1) $2x + y + (x+2y)y' = 0$ $y' = \frac{-(2x+y)}{x+2y}$

$2 + y' + (1+2y')y' + (x+2y)y'' = 0$, $2 - 2\frac{2x+y}{x+2y} + 2\left(\frac{2x+y}{x+2y}\right)^2 + (x+2y)y'' = 0$

$(x+2y)y'' = \frac{-2}{(x+2y)^2}\{(x+2y)^2 - (2x+y)(x+2y) + (2x+y)^2\}$

$= \frac{-2}{(x+2y)^2}(3x^2 + 3xy + 3y^2) = \frac{-6}{(x+2y)^2}$

$y'' = \frac{-6}{(x+2y)^3}$

(2) $2x + y = 0$ $y = -2x$ $x^2 - 2x^2 + 4x^2 = 3$ $x = \pm 1$ $y = \mp 2$

$x = -1$ のとき 極大値 $y = 2$

$x = 1$ のとき 極小値 $y = -2$

P.58

4.6 $f(x,y)=x^3-3axy+y^3$

(1) $f_x=3x^2-3ay$

$f_y=3y^2-3ax$

$x^2-ay=0$

$y^2-ax=0$

$(x-y)(x+y+a)=0$

$x=y=0$ $\quad y=-x-a$

$x=y=a$ $\quad x^2+ax+a^2=0$ ×

$(0,0),(a,a)$

$f_{xx}=6x \quad f_{yy}=6y \quad f_{xy}=-3a$

$D=36xy-9a^2$ とおくと $(0,0)$で $D<0$ 極値なし (a,a)で $D>0$

$a<0$ のとき (a,a)で $f_{xx}=6a<0$ 極大値 $-a^3$

$a>0$ のとき (a,a)で $f_{xx}=6a>0$ 極小値 $-a^3$

(2) $3x^2-3ay-(3ax-3y^2)y'=0$

$y'=\dfrac{x^2-ay}{ax-y^2}$ $\quad x^2-ay=0 \quad y=\dfrac{x^2}{a}$ $\quad x^3-3ax\dfrac{x^2}{a}+\dfrac{x^6}{a^3}=0$

$-2x^3+\dfrac{x^6}{a^3}=0$

$x=0 \quad \sqrt[3]{2}\,a$

$6x-3ay'-(3a-6yy')y'-(3ax-3y^2)y''=0$ $\quad y=0 \quad y=\sqrt[3]{4}\,a$

$y''=\dfrac{2x}{ax-y^2}$ $\quad \dfrac{2\sqrt[3]{2}\,a}{\sqrt[3]{2}\,a^2-2\sqrt[3]{2}\,a^2}=-\dfrac{2}{a}$

$0<a$ のとき $x=\sqrt[3]{2}\,a$で 極大値 $\sqrt[3]{4}\,a$

$a<0$ $\quad x=\sqrt[3]{2}\,a$で 極小値 $\sqrt[3]{4}\,a$

4.7 (1) $f(x,y)=\frac{1}{2}\{(x-1)^2+y^2\}+\sin\sqrt{x^2+y^2}$

$f_x=x-1+\dfrac{x}{\sqrt{x^2+y^2}}\cos\sqrt{x^2+y^2}$

$f_y=y+\dfrac{y}{\sqrt{x^2+y^2}}\cos\sqrt{x^2+y^2}$

$y\left(1+\dfrac{1}{\sqrt{x^2+y^2}}\cos\sqrt{x^2+y^2}\right)=0$ $\quad y=0$ or $\dfrac{1}{\sqrt{x^2+y^2}}\cos\sqrt{x^2+y^2}=-1$

$f_x=0$ に代入すると $x-1-x=0$ 不能

∴ $y=0$ $\quad$ ∴ $x-1+\dfrac{x}{|x|}\cos|x|=0$

$g(x)=x-1\pm\cos x$ とおくと $g'(x)=1\mp\sin x\geqq 0$ ∴ $g(x)$は増加

∴ $g(x)=0$ はただ1つの解 $x=0$ をもつ $\quad x\to+0$ のとき $x-1+\frac{x}{|x|}\cos|x|\to 0$

∴ $f_x=0,\ f_y=0$ の解は $(0,0)$のみである。

(2) $(\sin x)'=\cos x \quad (\sin x)''=-\sin x$

∴ $\sin x=x-\dfrac{x^2}{2}\sin(\theta x)$ $\quad 0<\theta<1$

p58

$$\frac{x^2}{2}\sin\theta x = x-\sin x \quad \therefore \ |x-\sin x| = \frac{x^2}{2}|\sin\theta x| \leqq \frac{x^2}{2}$$

$$\therefore \quad -\frac{x^2}{2} \leqq x-\sin x \leqq \frac{x^2}{2} \qquad \therefore \quad x \leqq \sin x + \frac{x^2}{2}$$

(3) (2) より

$$\sqrt{x^2+y^2} \leqq \sin\sqrt{x^2+y^2} + \frac{1}{2}(x^2+y^2)$$

$$\sqrt{x^2+y^2} - x + \frac{1}{2} \leqq \frac{1}{2}\{(x-1)^2+y^2\} + \sin\sqrt{x^2+y^2}$$

$$\sqrt{x^2+y^2} - x \geqq 0$$

$$\therefore \quad \frac{1}{2} \leqq \frac{1}{2}\{(x-1)^2+y^2\} + \sin\sqrt{x^2+y^2}$$

等号が成り立つのは $(0,0)$ のときのみ　極小値 $f(0,0)=\frac{1}{2}$

4.8　$x^2+y^2-2xy-2y-1=0 \qquad 2x-2y+(2y-2x-2)y'=0$

$y'=\dfrac{-(x-y)}{y-x-1} \qquad y'=0 \quad y=x=-\frac{1}{2}$

$1-y'+(y'-1)y'+(y-x-1)y''=0 \qquad y''=\dfrac{-1}{y-x-1}>0$

$\therefore$ $x=-\frac{1}{2}$ のとき 極小値 $y=-\frac{1}{2}$

4.9　$f(x,y)=x^2+ay+by^2+cy^3$

$f_x(x,y)=2x$ 　　　　$x=0$

$f_y(x,y)=a+2by+3cy^2$ 　　　　$3cy^2+2by+a=0$

$b^2-3ac \leqq 0$

$f_{xx}(x,y)=2$

$f_{xy}(x,y)=0$ 　　　　$D=4(b+3cy)>0 \qquad b+3cy>0$

$f_{yy}(x,y)=2b+6cy$

$b^2 \leqq 3ac;\ b<0,\ c=0;\ a\neq 0,\ b=c=0$

4.10　$f(x,y)=(x+y)^2+\dfrac{3(x+y)}{xy}=(x+y)^2+3\left(\dfrac{1}{x}+\dfrac{1}{y}\right)$

$f_x=2(x+y)-\dfrac{3}{x^2}$ 　　　　$y=\pm x \quad y=x \quad 4x^3-3=0 \quad x=y=\left(\frac{3}{4}\right)^{\frac{1}{3}}$

$f_y=2(x+y)-\dfrac{3}{y^2}$ 　　　　$y=-x$

$f_{xx}=2+\dfrac{6}{x^3}$

$f_{xy}=2$ 　　　　$D=f_{xx}f_{yy}-f_{xy}^2>0 \qquad f_{xx}>0$

$f_{yy}=2+\dfrac{6}{y^3}$

$\left(\left(\frac{3}{4}\right)^{\frac{1}{3}}, \left(\frac{3}{4}\right)^{\frac{1}{3}}\right)$ のとき 極小値 $9\sqrt[3]{\frac{4}{3}}$

P.59

§5. 偏微分の応用2（最大・最小）

5.1 (1) $x+y+z=9$ $f(x,y,z)=xyz$

$z=9-x-y$ $\therefore f=xy(9-x-y)=9xy-x^2y-xy^2$

$f_x=9y-2xy-y^2$ $y(9-2x-y)=0$ $x=y=0$

$f_y=9x-x^2-2xy$ $x(9-x-2y)=0$ $x=0$ $y=9$ $y=0$ $x=9$

$(0,0),(9,0),(0,9)\ (3,3)$ $-9+3y=0$ $y=x=3$

$f_{xx}=-2y$ $f_{xy}=9-2x-2y$ $f_{yy}=-2x$

$D=f_{xx}f_{yy}-f_{xy}^2=4xy-(9-2x-2y)^2$

$(0,0)$ のとき $D<0$. $(9,0)$, $(0,9)$ のとき $D<0$

$(3,3)$ のとき $D=36-9>0$ $f_{xx}=-6<0$

$(3,3)$ で極大値 27（最大値）

(2) $x^2+y^2=1$; $f=x^2+2xy+y^2$

$2x+2y-2\lambda x=0$ $y=(\lambda-1)x$ $y\{1-(\lambda-1)^2\}=0$

$2y+2x-2\lambda y=0$ $x=(\lambda-1)y$ $y=0$, $\lambda=0$, $\lambda=2$

$x=0$ $y=-x$ $x=y$

$(\pm\frac{1}{\sqrt2},\mp\frac{1}{\sqrt2})$ のとき 最小値 0 $x=\pm\frac{1}{\sqrt2}$, $y=\mp\frac{1}{\sqrt2}$ $x=y=\pm\frac{1}{\sqrt2}$

$(\pm\frac{1}{\sqrt2},\pm\frac{1}{\sqrt2})$ のとき 最大値 2

5.2 $x^2+y^2+z^2\leqq1$ $f=x^2+y^2-xy+yz+zx$

$x^2+y^2+z^2=1$ とおくと $z_x=-\frac{x}{z}$ $z_y=-\frac{y}{z}$

$f_x=2x-y+z+(y+x)(-\frac{x}{z})=0$ $2xz-yz+z^2-(xy+x^2)$ … $2yz-yz+z^2\cdot$ … $(2yx-x^2+zx)=0$

$f_y=2y-x+z+(x+y)(-\frac{y}{z})=0$ $x^2-y^2+z(y-x)=0$

$(x-y)(x+y-z)=0$

$y=x$ $x+z-\frac{2x^2}{z}=0$ $z^2+xz-2x^2=0$ $(z+2x)(z-x)=0$

$x=z$ $z=-2x$ $x=y=z=\frac{\pm1}{\sqrt3}$ $x=y=\frac{\pm1}{\sqrt6}$ $z=\mp\frac{2}{\sqrt6}$

$x+y=z$ $2x-y+z-x=0$ $z=y-x$

$2y-x+z-y=0$ $z=x-y$

$\therefore x=y=0$.

$(0,0)$ $(\frac{\pm1}{\sqrt3},\frac{\pm1}{\sqrt3},\frac{\pm1}{\sqrt3})$ $(\frac{\pm1}{\sqrt6},\frac{\pm1}{\sqrt6},\mp\frac{2}{\sqrt6})$

$f=\frac{1}{3}+\frac{1}{3}-\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1$ 最大値

$f=\frac{1}{6}+\frac{1}{6}-\frac{1}{6}-\frac{2}{6}-\frac{2}{6}=-\frac{1}{2}$ 最小値

P59

5.3 $x, y, z > 0 \qquad x+y+z=1 \qquad$ 底 2

$H(x,y,z) = -\{x\log_2 x + y\log_2 y + z\log_2 z\}$

$z = 1-x-y$ より $\quad H = -\{x\log_2 x + y\log_2 y + (1-x-y)\log_2(1-x-y)\}$

$H_x = -\log_2 x - \dfrac{1}{\log 2} + \log_2(1-x-y) - (1-x-y)\dfrac{-1}{1-x-y}\dfrac{1}{\log 2}$

$= \log_2 \dfrac{1-x-y}{x} \qquad \begin{cases} 1-x-y = x \\ 1-x-y = y \end{cases} \qquad x=y=z=\dfrac{1}{3}$

$H_y = \log_2 \dfrac{1-x-y}{y}$

$H_{xx} = -\dfrac{1}{x\log 2} - \dfrac{1}{(1-x-y)\log 2} \qquad H_{xy} = \dfrac{-1}{(1-x-y)\log 2}$

$H_{yy} = -\dfrac{1}{y\log 2} - \dfrac{1}{(1-x-y)\log 2}$

$D = H_{xx}H_{yy} - H_{xy}^2 = \dfrac{1}{(\log 2)^2}\left(\dfrac{1}{xy} + \left(\dfrac{1}{x}+\dfrac{1}{y}\right)\dfrac{1}{1-x-y}\right)$

$\left(\dfrac{1}{3}, \dfrac{1}{3}\right)$ で $D>0 \qquad H_{xx} < 0 \qquad \left(\dfrac{1}{3}, \dfrac{1}{3}\right)$ で最大値 $\log_2 3$

5.4 $\dfrac{x^2}{4} + y^2 \leqq 1 \qquad z = x + y + \sqrt{1 - \dfrac{x^2}{4} - y^2}$

$\dfrac{x^2}{4} + y^2 = r^2$ とおく $\quad |r| \leqq 1 \qquad r\dfrac{\partial r}{\partial x} = \dfrac{x}{4}$

$z_x = 1 - \dfrac{r}{\sqrt{1-r^2}}\dfrac{\partial r}{\partial x} = 1 - \dfrac{1}{\sqrt{1-r^2}}\dfrac{x}{4} \qquad r\dfrac{\partial r}{\partial y} = y$

$4\sqrt{1-r^2} = x \qquad x = 4y$

$z_y = 1 - \dfrac{r}{\sqrt{1-r^2}}\dfrac{\partial r}{\partial y} = 1 - \dfrac{1}{\sqrt{1-r^2}}y \qquad \sqrt{1-r^2} = y$

$4y^2 + y^2 = r^2 \qquad y = \pm\dfrac{1}{\sqrt{5}}r \qquad x = \pm\dfrac{4}{\sqrt{5}}r$

$\left(\dfrac{4}{\sqrt{5}}r, \dfrac{1}{\sqrt{5}}r\right)$ のとき $\quad z = \sqrt{5}r + \sqrt{1-r^2}$, 最大値 $\sqrt{5}$

$\left(-\dfrac{4}{\sqrt{5}}r, -\dfrac{r}{\sqrt{5}}\right)$ のとき $\quad z = -\sqrt{5}r + \sqrt{1-r^2}$, 最小値 $-\sqrt{5}$

5.5 $m, n, p > 0 \qquad x+y+z=a \qquad$ (5.4 と 5.5 は)

$V = x^m y^n z^p = x^m y^n (a-x-y)^p$

$V_x = \{m(a-x-y) - px\} x^{m-1} y^n (a-x-y)^{p-1}$

$V_y = \{n(a-x-y) - py\} x^m y^{n-1} (a-x-y)^{p-1}$

$ma - (m+p)x - my = 0 \qquad x = \dfrac{ma}{m+n+p}$

$na - nx - (n+p)y = 0 \qquad y = \dfrac{na}{m+n+p}$

$z = a - x - y = \dfrac{pa}{m+n+p}$

$\therefore V = m^m n^n p^p \dfrac{1}{(m+n+p)^{m+n+p}}$

P.59

5.6 (5.5を5.6に)

(x,y) が $x^2+y^2=1$ のとき $f(x,y)=ax^2+2bxy+cy^2$ の最大値、最小値

$\begin{cases} ax+by-\lambda x=0 \\ bx+cy-\lambda y=0 \end{cases}$ $\begin{cases} (a-\lambda)x+by=0 \\ bx+(c-\lambda)y=0 \end{cases}$ が $(0,0)$ 以外に解をもつのは

$\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix}=0$ この解を λ とすると

$ax^2+2bxy+cy^2-\lambda(x^2+y^2)=0$ $\therefore ax^2+2bxy+cy^2=\lambda$

2次曲線 $x^2+y^2=1$ と $ax^2+2bxy+cy^2=\lambda$ が共通点をもち、接線が同じとき最大値、最小値となる

5.7 (5.6を5.7に)

$0 \leqq x \leqq 1$ $y=e^x$ $y=ax+b$

$I(a,b)=\int_0^1 \{e^x-(ax+b)\}^2dx=\int_0^1 \{e^{2x}-2(ax+b)e^x+(ax+b)^2\}dx$

$=\left[\frac{1}{2}e^{2x}-2(ax+b)e^x+2ae^x+\frac{1}{3a}(ax+b)^3\right]_0^1$

$=\frac{1}{2}(e^2-1)-2(a+b)e+2b+2ae-2a+\frac{1}{3a}\{(a+b)^3-b^3\}$

$=\frac{1}{2}(e^2-1)-2be+2b-2a+\frac{1}{3}a^2+ab+b^2$

$I_a=-2+\frac{2}{3}a+b=0$ $-6+2a+3b=0$ $-6-4+4e-b=0$

$I_b=2-2e+a+2b=0$ $2-2e+a+2b=0$ $\begin{cases} b=4e-10 \\ a=-6e+18 \end{cases}$

5.8 $E=\sum_{i=1}^{5}\{y_i-(ax_i+b)\}^2=\sum_{i=1}^{5}y_i^2-2\sum_{i=1}^{5}y_i(ax_i+b)+\sum_{i=1}^{5}(ax_i+b)^2$

$E_a=-2\sum y_ix_i+2\sum(ax_i+b)x_i$

$E_b=-2\sum y_i+2\sum(ax_i+b)$

$\therefore a\sum x_i^2+b\sum x_i=\sum x_iy_i$ $a\sum x_i^2+5b\bar{x}=\sum x_iy_i$

$a\sum x_i+5b=\sum y_i$ $a\bar{x}+b=\bar{y}$

$a=\dfrac{\sum x_iy_i-5\bar{x}\bar{y}}{\sum x_i^2-5\bar{x}^2}$ $b=\dfrac{\bar{y}\sum x_i^2-\bar{x}\sum x_iy_i}{\sum x_i^2-5\bar{x}^2}$

p.59

5.9 (5.8を5.9に)

$V = a^2h = 4000 \qquad S = a^2 + 4ah$

$\therefore V = a^2 + \frac{16{,}000}{a} \qquad V_a = 2a - \frac{16{,}000}{a^2} \qquad a^3 - 8{,}000 = 0 \quad a = 20$

$\therefore a = 20 \quad h = 10$

5.10 (5.9を5.10に)

$x + y + z = a \qquad z = a - x - y$

$V = xyz = axy - x^2y - xy^2$

$V_x = ay - 2xy - y^2 \qquad y(a - 2x - y) = 0$

$V_y = ax - x^2 - 2xy \qquad x(a - x - 2y) = 0$

$(0, 0) \quad (0, a), \ (a, 0), \ \left(\frac{a}{3}, \frac{a}{3}\right)$

$V_{xx} = -2y \quad V_{yy} = -2x \quad V_{xy} = a - 2x - 2y$

$D = V_{xx}V_{yy} - V_{xy}^2 = 4xy - (a - 2x - 2y)^2$

$(0, 0), (0, a), (a, 0)$ のとき $D < 0$

$\left(\frac{a}{3}, \frac{a}{3}\right)$ のとき $D > 0 \quad V_{xx} < 0$

$\therefore \left(\frac{a}{3}, \frac{a}{3}\right)$ のとき最大値 $\frac{a^3}{27}$

P. 59

§ 6. 偏微分の応用（図形）

6.1 (1) $f(x,y) = x^3+y^3-3xy$

$f_x = 3x^2-3y$ $\quad x^2-y=0 \quad (0,0) \quad (1,1)$

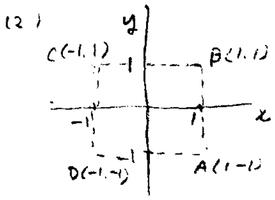
$f_y = 3y^2-3x$ $\quad y^2-x=0$

$f_{xx} = 6x,\ f_{yy} = 6y \quad f_{xy} = -3$

$D = 36xy-9$ $(0,0)$のとき $D<0$ $(1,1)$のとき $D>0$

$f_{xx}>0$. ∴ $(1,1)$で極小値 -1.

(2)



AB上で $f(x,y) = 1+y^3-3y \quad -1\leq y\leq 1$

BC上で $f(x,y) = x^3+1-3x \quad -1\leq x\leq 1$

CD上で $f(x,y) = -1+y^3+3y \quad -1\leq y\leq 1$

DA上で $f(x,y) = x^3-1+3x \quad -1\leq x\leq 1$

AB上で $\frac{df}{dy} = 3(y^2-1)<0$ 減少 Max; $f(A)=3$ min: $f(B)=-1$

BC上で $\frac{df}{dx} = 3(x^2-1)<0$ 減少 Max $f(C)=3$ min $f(B)=-1$

CD上で $\frac{df}{dy} = 3(y^2+1)>0$ 増加 Max; $f(C)=3$ min $f(D)=-5$

DA上で $\frac{df}{dx} = 3(x^2+1)>0$ 増加 Max: $f(A)=3$ min $f(D)=-5$

(3) $|x|<1, |y|<1$ において $f(x,y)$ が最大値または最小値をとれば それは極大値または極小値となるが, (1)より極値は $f(1,1)$ のみである. 故に 最大値は $f(-1,1)=f(1,-1)=3$

最小値は $f(-1,-1)=-5$

6.2 $P_i(x_i,y_i) \quad P(x,y)$

$V = \sum_{i=1}^{n}\{(x_i-x)^2+(y_i-y)^2\}$

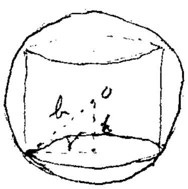
$V_x = -2\sum_{i=1}^{n}(x_i-x) \quad V_y = -2\sum_{i=1}^{n}(y_i-y)$ $\quad x = \frac{1}{n}\sum x_i = \bar{x}$

$y = \frac{1}{n}\sum y_i = \bar{y}$

$V_{xx} = 2n \quad V_{yy} = 2n \quad V_{xy} = 0$

$D = V_{xx}V_{yy}-V_{xy}^2 = 4n^2>0$ ∴ 最小値 $V(\bar{x},\bar{y})$

6.3



$r^2+h^2=b^2 \quad 0\leq b\leq R$

$V = 2\pi r^2 h$, $S = 2\pi r^2+2\pi rh$

$F = \frac{S}{V} = \frac{1}{h}+\frac{1}{r}$

$\frac{-1}{h^2}+2\lambda h=0 \quad -\frac{1}{r^2}+2\lambda r=0$

$h = r = \sqrt[3]{\frac{1}{2\lambda}} = \frac{b}{\sqrt{2}}$

$\frac{2}{(2\lambda)^{\frac{2}{3}}} = b^2 \quad \left(\frac{2}{b^2}\right)^3 = (2\lambda)^2$

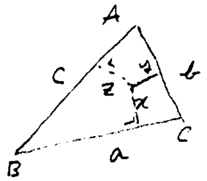
$\frac{2}{b^6} = \lambda^2 \quad \lambda = \frac{\sqrt{2}}{b^3}$

∴ $F = \frac{2\sqrt{2}}{b}$.

P 59. 60

$\therefore$ $b=a$ のとき最小となり　底面の半径：高さ $=1:2$

6.4



$$ax+by+cz=k\ (一定) \qquad 2S=k$$

$$V=xyz=xy\frac{1}{c}(k-ax-by)$$

$$=\frac{1}{c}(kxy-ax^2y-bxy^2)$$

$$V_x=\frac{1}{c}(ky-2axy-by^2)=\frac{y}{c}(k-2ax-by)=0$$

$$V_y=\frac{1}{c}(kx-ax^2-2bxy)=\frac{x}{c}(k-ax-2by)=0$$

$$(0,0)\quad\left(\frac{k}{a},0\right)\quad\left(0,\frac{k}{b}\right)\quad\left(\frac{k}{3a},\frac{k}{3b}\right)$$

$$V_{xx}=-\frac{2a}{c}y \qquad V_{yy}=-\frac{2b}{c}x \qquad V_{xy}=\frac{1}{c}(k-2ax-2by)$$

$$D=V_{xx}V_{yy}-V_{xy}^2=\frac{1}{c^2}\{4abxy-(k-2ax-2by)^2\}$$

$(0,0)$, $\left(\frac{k}{a},0\right)$　$\left(0,\frac{k}{b}\right)$ のとき　$D<0$

$\left(\frac{k}{3a},\frac{k}{3b}\right)$ のとき　$D>0$　$V_{xx}<0$

$$\therefore V(x,y)=\frac{1}{c}\frac{k^2}{3a\cdot 3b}\cdot\frac{k}{3}=\frac{k^3}{27abc}=\frac{8S^3}{27abc}$$

6.5

(1) $\begin{cases} x+y+z=6 \\ xy+yz+zx=9 \end{cases}$　$2(xy+yz+zx)=18$

$$xy+(x+y)\{6-(x+y)\}=9$$

$$(x+y)^2-6(x+y)-xy+9=0$$

$$y^2+(x-6)y+x^2-6x+9=0 \qquad y^2+(x-6)y+(x-3)^2=0$$

$$\therefore (x-6)^2-4(x-3)^2\geqq 0 \qquad (x-6-2x+6)(x-6+2x-6)\geqq 0$$

$$-x(3x-12)\geqq 0 \qquad x(x-4)\leqq 0 \qquad 0<x\leqq 4$$

(2)　$x+y+z=6$ より　$y+z=6-x$

$xy+yz+zx=9$ より　$x(6-x)+yz=9$　$\therefore yz=9-6x+x^2=(x-3)^2$

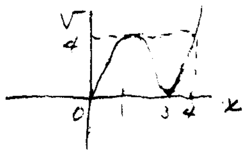
$$\therefore V=xyz$$

$$=x(x-3)^2$$

$$V'=(x-3)^2+2x(x-3)=(x-3)(3x-3)=3(x-3)(x-1)$$

$\therefore V$ は　$0<x\leqq 1$ で増加　$1<x\leqq 3$ で減少　$3<x\leqq 4$ で増加

$\therefore V(1)=4$　$V(4)=4$



$\therefore$ 最大値　$(1,1,4)(1,4,1)$ or $(4,1,1)$

で 4 //

P.60

6.6 $xy+yz+zx=\frac{1}{2}$

$V=xyz=xy\frac{1}{x+y}\left(\frac{1}{2}-xy\right)=\frac{1}{x+y}\left(\frac{1}{2}xy-x^2y^2\right)$

$V_x=\frac{1}{x+y}\left(\frac{1}{2}y-2xy^2\right)-\frac{1}{(x+y)^2}\left(\frac{1}{2}xy-x^2y^2\right)=\frac{1}{(x+y)^2}\left(-x^2y^2+\frac{1}{2}y^2-2xy^3\right)$

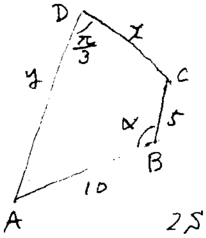
$V_y=\frac{1}{(x+y)^2}\left(-x^2y^2+\frac{1}{2}x^2-2x^3y\right)$

$y^2\left(-x^2+\frac{1}{2}-2xy\right)=0$ $(0,0)$ $\left(\frac{1}{\sqrt6},\frac{1}{\sqrt6}\right)$

$x^2\left(-y^2+\frac{1}{2}-2xy\right)=0$

$\therefore V\left(\frac{1}{\sqrt3},\frac{1}{\sqrt3}\right)=\frac{1}{6\sqrt6}$ 最大値

6.7



$AC^2=100+25-100\cos\alpha$

$AC^2=x^2+y^2-2xy\cos\frac{\pi}{3}$

$125-100\cos\alpha=x^2+y^2-xy$

$\therefore$ $100\sin\alpha\frac{\partial\alpha}{\partial x}=2x-y$ $100\sin\alpha\frac{\partial\alpha}{\partial y}=2y-x$

$2S=xy\sin\frac{\pi}{3}+50\sin\alpha$

$=\frac{\sqrt3}{2}xy+50\sin\alpha$

$\frac{\partial(2S)}{\partial x}=\frac{\sqrt3}{2}y+50\cos\alpha\frac{\partial\alpha}{\partial x}=\frac{\sqrt3}{2}y+\frac{50\cos\alpha}{100\sin\alpha}(2x-y)$

$=\frac{\sqrt3}{2}y+\frac{1}{2}\cot\alpha\,(2x-y)$

$\frac{\partial(2S)}{\partial y}=\frac{\sqrt3}{2}x+\frac{1}{2}\cot\alpha(2y-x)$

$\begin{cases} x\cot\alpha+y\left(\frac{\sqrt3}{2}-\frac{1}{2}\cot\alpha\right)=0 \\ y\cot\alpha+x\left(\frac{\sqrt3}{2}-\frac{1}{2}\cot\alpha\right)=0 \end{cases}$ これが $(0,0)$ 以外に解をもつ

$\cot^2\alpha-\left(\frac{\sqrt3}{2}-\frac{1}{2}\cot\alpha\right)^2=0$ $\cot^2\alpha+\frac{2}{\sqrt3}\cot\alpha-1=0$

$\cot\alpha=-\frac{1}{\sqrt3}\pm\frac{2}{\sqrt3}$ $\frac{1}{\sqrt3}$, $-\sqrt3$ $\alpha=\frac{\pi}{3}$, $\frac{5}{6}\pi$

i) $\alpha=\frac{\pi}{3}$ のとき $75=x^2+y^2-xy$ $2S=\frac{\sqrt3}{2}xy+25\sqrt3$

$\frac{\sqrt3}{2}y+\lambda(2x-y)=0$ $2\lambda x+\left(\frac{\sqrt3}{2}-\lambda\right)y=0$

$\frac{\sqrt3}{2}x+\lambda(2y-x)=0$ $\left(\frac{\sqrt3}{2}-\lambda\right)x+2\lambda y=0$ これが $(0,0)$ 以外の解をもつ

$\left(\frac{\sqrt3}{2}-\lambda\right)^2-4\lambda^2=0$ $\left(\frac{\sqrt3}{2}-3\lambda\right)\left(\frac{\sqrt3}{2}+\lambda\right)=0$ $\lambda=\frac{1}{2\sqrt3}$ or $-\frac{\sqrt3}{2}$

$\lambda=\frac{1}{2\sqrt3}$ のとき $\frac{x}{\sqrt3}+\frac{y}{\sqrt3}=0$ $y=-x$ $x,y>0$ より 不適

$\lambda=\frac{-\sqrt3}{2}$ のとき $-x+y=0$ $y=x$ $x^2=75$ $x=y=5\sqrt3$

P.60

$$2S = 25\sqrt{3} + \frac{\sqrt{3}}{2}25\ 3 = \frac{125\sqrt{3}}{2} \qquad S = \frac{125\sqrt{3}}{4}$$

ii) $\alpha = \frac{5}{6}\pi$ のとき

$$125 + 50\sqrt{3} = x^2 + y^2 - xy \qquad 2S = \frac{\sqrt{3}}{2}xy + 25$$

$$\left.\begin{aligned} &\frac{\sqrt{3}}{2}y + \lambda(2x - y) = 0 \qquad 2\lambda x + \left(\frac{\sqrt{3}}{2} - \lambda\right)y = 0 \\ &\frac{\sqrt{3}}{2}x + \lambda(2y - x) = 0 \qquad \left(\frac{\sqrt{3}}{2} - \lambda\right)x + 2\lambda y = 0 \end{aligned}\right) \ (0,0)\text{以外の解をもつ}$$

$$\left(\frac{\sqrt{3}}{2} - \lambda\right)^2 - 4\lambda^2 = 0 \qquad \lambda = \frac{1}{2\sqrt{3}}\ ,\ -\frac{\sqrt{3}}{2}$$

$\lambda = \frac{1}{2\sqrt{3}}$ のとき $\quad y = -x \quad \times$

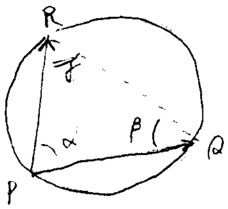
$\lambda = -\frac{\sqrt{3}}{2}$ のとき $\quad y = x$

$$x^2 = 125 + 50\sqrt{3} = 25(5 + 2\sqrt{3}) \qquad x = y = 5\sqrt{5 + 2\sqrt{3}}$$

$$2S = \frac{\sqrt{3}}{2}25(5 + 2\sqrt{3}) + 25 = 100 + \frac{125\sqrt{3}}{2}$$

$\therefore S = 50 + \frac{125\sqrt{3}}{4}$ 最大値.

6.8



$$\frac{PQ}{\sin\gamma} = \frac{PR}{\sin\beta} = \frac{QR}{\sin\alpha} = 2a \qquad \alpha + \beta + \gamma = \pi$$

$\therefore PQ = 2a\sin(\alpha + \beta) \quad PR = 2a\sin\beta \quad QR = 2a\sin\alpha$

$\overrightarrow{PQ} \cdot \overrightarrow{PR} = PQ \cdot PR \cos\alpha = 4a^2\cos\alpha\sin\beta\sin(\alpha + \beta) = f$ とおく

$$f_\alpha = 4a^2\{-\sin\alpha\sin\beta\sin(\alpha + \beta) + \cos\alpha\sin\beta\cos(\alpha + \beta)\}$$
$$= 4a^2\sin\beta\cos(2\alpha + \beta)$$
$$f_\beta = 4a^2\{\cos\alpha\cos\beta\sin(\alpha + \beta) + \cos\alpha\sin\beta\cos(\alpha + \beta)\}$$
$$= 4a^2\cos\alpha\sin(2\beta + \alpha)$$

$\alpha \neq 0\ \ \pi \qquad \therefore \cos(2\alpha + \beta) = 0 \quad \sin(2\beta + \alpha) = 0$

$$\begin{cases} 2\alpha + \beta = \frac{\pi}{2} \\ \alpha + 2\beta = \pi \end{cases} \qquad \begin{aligned} 2\alpha + \beta &= \frac{3}{2}\pi \\ \alpha + 2\beta &= \pi\ .\ 2\pi \end{aligned}$$

$3\beta = \frac{2}{2}\pi \qquad 3\beta = \frac{1}{2}\pi \qquad 3\beta = \frac{5}{2}\pi$

$\beta = \frac{\pi}{2} \quad \alpha = 0 \qquad \beta = \frac{\pi}{6} \quad \alpha = \frac{2}{3}\pi \qquad \beta = \frac{5}{6}\pi \quad \alpha = \frac{\pi}{3}$

$$(\alpha, \beta) = \left(0, \frac{\pi}{2}\right) \quad \left(\frac{2}{3}\pi, \frac{\pi}{6}\right) \quad \left(\frac{\pi}{3}, \frac{5}{6}\pi\right)$$

$\left(\frac{2}{3}\pi, \frac{\pi}{6}\right)$ のとき $\quad \overrightarrow{PQ} \cdot \overrightarrow{PR} = 4a^2\left(-\frac{1}{2}\right)\frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{2}a^2$

$\left(\frac{\pi}{3}\ \frac{5}{6}\pi\right)$ 〃 $\quad \overrightarrow{PQ} \cdot \overrightarrow{PR} = 4a^2\frac{1}{2} \cdot \frac{1}{2}\ \frac{1}{2} = \frac{1}{2}a^2$

$\angle\alpha = \frac{2}{3}\pi$ のとき $\ -\frac{1}{2}a^2$ 最小

p60

6.9 $x+y+z=k$

$S=2(xy+yz+zx)=2xy+2z(x+y)=2xy+2(k-(x+y))(x+y)$

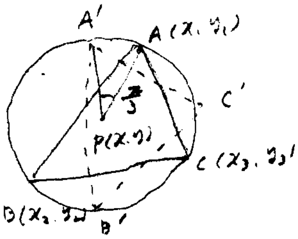
$=2xy+2k(x+y)-2(x+y)^2$

$S_x=2y+2k-4(x+y)=0$ $\qquad x=y=\frac{k}{3}$

$S_y=2x+2k-4(x+y)=0$

$S=\frac{2}{3}k^2$

6.10



$A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$

$A'(x_1',y_1')$, $B'(x_2',y_2')$, $C'(x_3',y_3')$ とおくと

$$\begin{pmatrix} x_1'-x \\ y_1'-y \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} \begin{pmatrix} x_1-x \\ y_1-y \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} (x_1-x)-\sqrt{3}(y_1-y) \\ \sqrt{3}(x_1-x)+(y_1-y) \end{pmatrix}$$

$$2\triangle AA'P = \begin{vmatrix} x_1-x & x_1'-x \\ y_1-y & y_1'-y \end{vmatrix} = \begin{vmatrix} x_1-x, & \frac{1}{2}(x_1-x)-\frac{\sqrt{3}}{2}(y_1-y) \\ y_1-y, & \frac{\sqrt{3}}{2}(x_1-x)+\frac{1}{2}(y_1-y) \end{vmatrix}$$

$$= \begin{vmatrix} x_1-x, & -\frac{\sqrt{3}}{2}(y_1-y) \\ y_1-y, & \frac{\sqrt{3}}{2}(x_1-x) \end{vmatrix} = \frac{\sqrt{3}}{2}\{(x_1-x)^2+(y_1-y)^2\}$$

同様に $\triangle BB'P$, $\triangle CC'P$ も求められる

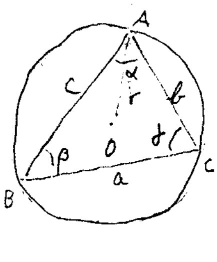
$\therefore I=\triangle AA'P+\triangle BB'P+\triangle CC'P=\frac{\sqrt{3}}{4}\{(x_1-x)^2+(x_2-x)^2+(x_3-x)^2+(y_1-y)^2+(y_2-y)^2+(y_3-y)^2\}$

$I_x=\frac{\sqrt{3}}{2}\{-(x_1-x)-(x_2-x)-(x_3-x)\}=0$ $\qquad x=\frac{1}{3}(x_1+x_2+x_3)$

$I_y=\frac{\sqrt{3}}{2}\{-(y_1-y)-(y_2-y)-(y_3-y)\}=0$ $\qquad y=\frac{1}{3}(y_1+y_2+y_3)$

$\therefore P\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$ 重心のとき最小となる。

6.11



$b=2r\sin\beta$ $\quad c=2r\sin\gamma$ $\qquad \alpha+\beta+\gamma=\pi$

$S=\frac{1}{2}bc\sin\alpha=2r^2\sin\alpha\sin\beta\sin\gamma$

$=2r^2\sin\alpha\sin\beta\sin(\alpha+\beta)$

$S_\alpha=2r^2\sin\beta\sin(2\alpha+\beta)=0$

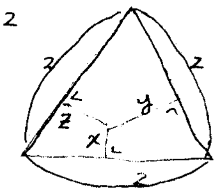
$S_\beta=2r^2\sin\alpha\sin(\alpha+2\beta)=0$

$2\alpha+\beta=\pi$ $\quad \alpha+2\beta=\pi$ $\qquad \alpha=\beta=\frac{\pi}{3}$

P60

∴ 三角形が正三角形のとき最大となる。

6.12



$x+y+z=\frac{1}{2}\cdot 2\sqrt{3}=\sqrt{3}$

$\therefore x+y+z=\sqrt{3}$

$I=x^2+y^2+z^2-xy-yz-zx$

$=x^2+y^2+\{\sqrt{3}-(x+y)\}^2-xy-\{\sqrt{3}-(x+y)\}(x+y)$

$=x^2+y^2+2(x+y)^2-3\sqrt{3}(x+y)-xy+3$

$I_x=2x+4(x+y)-3\sqrt{3}-y=6x+3y-3\sqrt{3}$ $\quad 2x+y-\sqrt{3}=0$

$I_y=2y+4(x+y)-3\sqrt{3}-x=3x+6y-3\sqrt{3}$ $\quad x+2y-\sqrt{3}=0$

$x=y=\frac{\sqrt{3}}{3}$

$I_{xx}=6 \quad I_{yy}=6 \quad I_{xy}=3 \quad D=36-9>0 \quad I_{xx}>0$

$\therefore\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ のとき最小値 0

P.60　§7 テーラー展開　§8 総合問題

7.1 (1) $f(x,y)=1/\sqrt{1-(x^2+y^2)} = \{1-(x^2+y^2)\}^{-\frac{1}{2}}$

$$=1+\left(-\frac{1}{2}\right)\{-(x^2+y^2)\}+\frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\{-(x^2+y^2)\}^2+\frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\{-(x^2+y^2)\}^3+\cdots$$

$$=1+\frac{1}{2}(x^2+y^2)+\frac{1\cdot3}{2!}\left(\frac{x^2+y^2}{2}\right)^2+\frac{1\cdot3\cdot5}{3!}\left(\frac{x^2+y^2}{2}\right)^3+\cdots+\frac{1\cdot3\cdots(2n-1)}{n!}\left(\frac{x^2+y^2}{2}\right)^n$$

(2) $f(x,y)=e^{ax}\cos by$

$$=\left(1+ax+\frac{a^2x^2}{2!}+\frac{a^3x^3}{3!}+\cdots+\frac{a^nx^n}{n!}+\cdots\right)\left(1-\frac{b^2}{2!}y^2+\frac{b^4}{4!}y^4-\cdots+\frac{(-1)^nb^{2n}}{(2n)!}y^{2n}+\cdots\right)$$

$$=1+ax+\frac{1}{2!}(a^2x^2-b^2y^2)+\frac{1}{3!}(a^3x^3-3ab^2xy^2)+\cdots+R_n$$

$$R_n=\sum_{r=0}^{[\frac{n}{2}]}\frac{(-1)^r a^{n-2r}b^{2r}}{(n-2r)!(2r)!}x^{n-2r}y^{2r}=\frac{1}{n!}\sum_{r=0}^{[\frac{n}{2}]}(-1)^r{}_nC_{2r}\,a^{n-2r}b^{2r}x^{n-2r}y^{2r}$$

7.2 $T=2\pi\sqrt{\frac{l}{g}}$　∴ $\log T=\log 2\pi+\frac{1}{2}\log l-\frac{1}{2}\log g$

$$\frac{dT}{T}=\frac{dl}{2l}-\frac{dg}{2g}$$

∴ $\frac{\Delta T}{T}\fallingdotseq\frac{1}{2}\left(\frac{\Delta l}{l}-\frac{\Delta g}{g}\right)$

8.1 (1) $\log\sqrt{x^2+y^2}=\tan^{-1}\frac{y}{x}$　$\frac{1}{2}\log(x^2+y^2)=\tan^{-1}\frac{y}{x}$

$$\frac{x}{x^2+y^2}+\frac{y}{x^2+y^2}\frac{dy}{dx}=\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)+\frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x}\frac{dy}{dx}$$

$$=\frac{-y}{x^2+y^2}+\frac{x}{x^2+y^2}\frac{dy}{dx}$$

$x+y=(x-y)\frac{dy}{dx}$　$\frac{dy}{dx}=\frac{x+y}{x-y}$

(2) $1+y'=(1-y')y'+(x-y)y''$　$y''=\frac{1}{x-y}(1+y'^2)=\frac{1}{x-y}\left\{\frac{(x-y)^2+(x+y)^2}{(x-y)^2}\right\}$

$$=\frac{2(x^2+y^2)}{(x-y)^3}$$

∴ $\frac{d^2y}{dx^2}=\frac{2(x^2+y^2)}{(x-y)^3}$

(3) $x=r\cos\theta$　$y=r\sin\theta$ とおくと $\log r=\tan^{-1}(\tan\theta)$

∴ $r=e^{\theta}$