IDENTIFICATION OF THE ABSENT SPECTRAL GAPS IN A CLASS OF GENERALIZED KRONIG-PENNEY HAMILTONIANS

HIROAKI NIIKUNI
(DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY)

In this talk we study the spectral gaps of the one-dimensional Schrödinger operators with particular periodic point interactions. We fix $\kappa \in (0,\pi) \cup (\pi, 2\pi)$. Let

$$\Gamma_1 = 2\pi\mathbf{Z}, \quad \Gamma_2 = \{\kappa\} + 2\pi\mathbf{Z}, \quad \Gamma = \Gamma_1 \cup \Gamma_2.$$

For $\theta_1, \theta_2 \in [-\pi/2, \pi/2)$ and $A_1, A_2 \in SO(2)\backslash\{\pm I\}$, we define the operator $H = H(\theta_1, \theta_2, A_1, A_2)$ in $L^2(\mathbf{R})$ as follows.

$$(Hy)(x) = -\frac{d^2}{dx^2}y(x), \quad x \in \mathbf{R}\setminus\Gamma,$$

$$\mathrm{Dom}(H) = \left\{ y \in H^2(\mathbf{R}\setminus\Gamma) \;\middle|\; \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = e^{i\theta_j} A_j \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix}, \quad x \in \Gamma_j, \quad j = 1, 2 \right\}.$$

Since $A_j \in SO(2)\setminus\{\pm I\}$, we can write the elements of A_j as

$$A_j = \begin{pmatrix} \cos\alpha_j & -\sin\alpha_j \\ \sin\alpha_j & \cos\alpha_j \end{pmatrix}, \quad \alpha_j \in (-\pi, 0) \cup (0, \pi).$$

The operator H is self-adjoint. Since the set $\sigma(H(\theta_1, \theta_2, A_1, A_2))$ is independent of θ_1 and θ_2, we hereafter discuss only the case where

$$\theta_1 = \theta_2 = 0.$$

Next, we define the spectral gaps of H. To this end, we consider the equation

$$(1) \qquad \begin{cases} -y''(x,\lambda) = \lambda y(x,\lambda), \quad x \in \mathbf{R}\setminus\Gamma \\ \begin{pmatrix} y(x+0,\lambda) \\ y'(x+0,\lambda) \end{pmatrix} = A_j \begin{pmatrix} y(x-0,\lambda) \\ y'(x-0,\lambda) \end{pmatrix} \text{ for } x \in \Gamma_j, \;\; j = 1, 2, \end{cases}$$

where λ is a real parameter. This equation has two solutions $y_1(x,\lambda)$ and $y_2(x,\lambda)$ which are uniquely determined by the initial conditions

$$y_1(+0,\lambda) = 1, \quad y_1'(+0,\lambda) = 0,$$

and

$$y_2(+0,\lambda) = 0, \quad y_2'(+0,\lambda) = 1,$$

respectively. We introduce the discriminant $D(\lambda)$ of the equation (1):

$$(2) \qquad D(\lambda) = y_1(2\pi+0,\lambda) + y_2'(2\pi+0,\lambda).$$

Let λ_j be the $(j+1)$st zero of $D(\cdot)^2 - 4$. Then we have

$$\lambda_0 < \lambda_1 \le \lambda_2 < \lambda_3 \le \lambda_4 < \cdots < \lambda_{2k-1} \le \lambda_{2k} < \cdots \to \infty.$$

We define

$$B_j = [\lambda_{2j-2}, \lambda_{2j-1}], \quad G_j = (\lambda_{2j-1}, \lambda_{2j}).$$

Then we derive

$$\sigma(H) = \bigcup_{j=1}^{\infty} B_j.$$

The open interval G_j is called the j-th gap of the spectrum of H, the closed interval B_j the j-th band. The aim of this study is to determine whether or not the j-th gap is absent for a given $j \in \mathbf{N}$. Throughout this talk we use the notations

$$a \equiv b \text{ if } a - b \in \pi\mathbf{Z}, \qquad a \not\equiv b \text{ if } a - b \notin \pi\mathbf{Z}$$

for a, $b \in \mathbf{R}$. For convenience we adopt the following classification of the parameters α_1 and α_2.

(I) $\alpha_1 - \alpha_2 \not\equiv 0, \quad \alpha_1 + \alpha_2 \not\equiv 0.$

(II) $\alpha_1 + \alpha_2 \equiv 0.$

(III) $\alpha_1 - \alpha_2 \equiv 0, \quad \alpha_1 + \alpha_2 \not\equiv 0.$

Our main results are the following three theorems.

Theorem 1. *If the condition* (I) *holds, then*

$$G_j \neq \emptyset \quad \text{for all} \quad j \in \mathbf{N}.$$

Theorem 2. *Suppose that* (II) *is valid.*
(1) *Let* $\kappa/\pi \notin \mathbf{Q}$. *Then we have*

$$\{j \in \mathbf{N} | \quad G_j = \emptyset\} = \{3\}.$$

(2) *If* $\kappa/2\pi = q/p$, $(p, q) \in \mathbf{N}^2$, $\gcd(p, q) = 1$, *then*

$$\{j \in \mathbf{N} | \quad G_j = \emptyset\} = \{3\} \cup \{pk + 1 \ | \ k \in \mathbf{N}\}.$$

Theorem 3. *Assume that* (III) *is valid. We put* $\eta_j = \pi^2 j^2 / 4(\pi - \kappa)^2$ *for* $j \in \mathbf{N}$. *Then it holds that*

$$\bigcup_{k=1}^{\infty} B_k \cap B_{k+1} = \left\{ \eta_j \middle| -2\left(\sqrt{\eta_j} + \frac{1}{\sqrt{\eta_j}}\right)^{-1} \cot \kappa \sqrt{\eta_j} = \tan \alpha_1 \text{ and } j \in \mathbf{N} \right\}.$$

References

[1] Chernoff, R. P., and Hughes, J. R., A new class of point interactions in one dimension, J. Funct. Anal., **111**(1993), pp. 97–117.

[2] Gan, S. and Zhang, M., Resonance pockets of Hill's equations with two-step potentials, SIAM J. Math. Anal., **32**(2000), No.3, pp.651–664.

[3] Gesztesy, F., Holden, H., and, Kirsch, W., On energy gaps in a new type of analytically solvable model in quantum mechanics, J. Math. Anal. Appl. **222** (1988), pp. 9–29.

[4] Gesztesy, F., and Kirsch, W., One-dimensional Schrödinger operators with interactions singular on a disctete set, J. Reine Angew. Math., **362** (1985), pp. 28–50.

[5] Hughes, R. J., Generalized Kronig-Penney Hamiltonians, J. Math. Anal. Appl., **222** (1998), no.1, pp. 151-166.

[6] Kronig, R., and Penney, W., Quantum mechanics in crystal lattices, Proc. Royal. Soc. London, **130** (1931), pp. 499–513.

[7] Šeba, P., The generalized point interaction in one dimension, Czech J. Phys. B **36** (1986), pp. 667–673.

[8] Yoshitomi, K., Spectral gaps of the one-dimensional Schrödinger operators with periodic point interactions, to appear in Hokkaido Math. J.