微分積分 I

p. 103　第 3 章　練習問題 II-B

1. (1) $\displaystyle\int \frac{dx}{\sqrt{x^2+4x+5}} = \int \frac{dx}{\sqrt{(x+2)^2+1}}$. $x+2=t$ とおくと $dx=dt$.

$\displaystyle\therefore$ 与式 $\displaystyle= \int \frac{dt}{\sqrt{t^2+1}} = \log|t+\sqrt{t^2+1}| = \log|x+2+\sqrt{x^2+4x+5}|$

(2) $\displaystyle\int \frac{dx}{\sqrt{3-2x-x^2}} = \int \frac{dx}{\sqrt{4-(x+1)^2}}$. $x+1=t$ とおくと $dx=dt$.

$\therefore$ 与式 $\displaystyle= \int \frac{dt}{\sqrt{4-t^2}} = \mathrm{Sin}^{-1}\frac{t}{2} = \mathrm{Sin}^{-1}\frac{x+1}{2}$

(3) $\displaystyle\int \frac{2x+3}{x^2+2x+2}dx = \int \frac{2x+2}{x^2+2x+2}dx + \int \frac{1}{x^2+2x+2}dx.$

第 1 項は $\dfrac{f'}{f}$ 第 2 項は $x^2+2x+2=(x+1)^2+1$ として $x+1=t$ とおくと $dx=dt$ だから

与式 $\displaystyle= \int \frac{(x^2+2x+2)'}{x^2+2x+2}dx + \int \frac{1}{t^2+1}dt = \log(x^2+2x+2) + \mathrm{Tan}^{-1}t = \log(x^2+2x+2) + \mathrm{Tan}^{-1}(x+1)$

(4) $\displaystyle\int x\log(x+1)dx = \frac{x^2}{2}\log(x+1) - \int \frac{x^2}{2}\cdot\frac{1}{x+1}dx = \frac{x^2}{2}\log(x+1) - \frac{1}{2}\int\left(x-1+\frac{1}{x+1}\right)dx$

$\displaystyle= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\left\{\frac{x^2}{2} - x + \log(x+1)\right\} = \frac{x^2-1}{2}\log(x+1) - \frac{x^2}{4} + \frac{x}{2}.$

2. (1) $x=\tan\theta$ とおくと $x^2+1=\tan^2\theta+1=\dfrac{1}{\cos^2\theta}$. よって $\dfrac{1}{(x^2+1)^{\frac{3}{2}}} = (\cos^2\theta)^{\frac{3}{2}} = \cos^3\theta.$

$dx = \dfrac{1}{\cos^2\theta}d\theta.$

x	0	$\to$	$\sqrt{3}$
θ	0	$\to$	$\frac{\pi}{3}$

$\therefore$ 与式 $\displaystyle= \int_0^{\frac{\pi}{3}} \cos^3\theta\cdot\frac{1}{\cos^2\theta}d\theta = \int_0^{\frac{\pi}{3}}\cos\theta d\theta = [\sin\theta]_0^{\frac{\pi}{3}} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$

(2) 半角の公式より $\dfrac{1+\cos x}{2} = \cos^2\dfrac{x}{2}$ よって $1+\cos x = 2\cos^2\dfrac{x}{2}$. $\therefore$ 与式 $\displaystyle= \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\sqrt{2\cos^2\frac{x}{2}}dx = \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\sqrt{2}\cos\frac{x}{2}dx$

$\displaystyle= \sqrt{2}\left[2\sin\frac{x}{2}\right]_{\frac{\pi}{2}}^{\frac{2\pi}{3}} = 2\sqrt{2}\left(\sin\frac{\pi}{3}-\sin\frac{\pi}{4}\right) = 2\sqrt{2}\left(\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\right) = \sqrt{6}-2.$

(3) $x=a\sin\theta$ とおくと $a^2-x^2=a^2-a^2\sin^2\theta=a^2(1-\sin^2\theta)=a^2\cos^2\theta.$

よって $\sqrt{(a^2-x^2)^3} = \sqrt{(a^2\cos^2\theta)^3} = a^3\cos^3\theta$. $dx = a\cos\theta d\theta.$

x	0	$\to$	a
θ	0	$\to$	$\frac{\pi}{2}$

$\therefore$ 与式 $\displaystyle= \int_0^{\frac{\pi}{2}} a^3\cos^3\theta\cdot a\cos\theta d\theta = a^4\int_0^{\frac{\pi}{2}}\cos^4\theta d\theta = a^4\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi a^4}{16}.$

(4) $\tan^2 x+1 = \dfrac{1}{\cos^2 x}$ より $\tan^2 x = \dfrac{1}{\cos^2 x} - 1$. よって

与式 $\displaystyle= \int_0^{\frac{\pi}{4}}\tan\cdot\tan^2 x dx = \int_0^{\frac{\pi}{4}}\tan x\left(\frac{1}{\cos^2 x}-1\right)dx = \int_0^{\frac{\pi}{4}}\tan\cdot\frac{1}{\cos^2 x}dx - \int_0^{\frac{\pi}{4}}\tan x dx.$

第 1 項は $\tan x = t$ とおくと $\dfrac{1}{\cos^2 x}dx = dt.$

x	0	$\to$	$\frac{\pi}{4}$
t	0	$\to$	1

第 2 項は $\tan x = \dfrac{\sin x}{\cos x}$ として $\cos x = u$ とおくと $-\sin x dx = du.$

x	0	$\to$	$\frac{\pi}{4}$
t	1	$\to$	$\frac{\sqrt{2}}{2}$

$\therefore$ 与式 $\displaystyle= \int_0^1 tdt - \int_1^{\frac{\sqrt{2}}{2}}\frac{-du}{u} = \left[\frac{t^2}{2}\right]_0^1 + [\log u]_1^{\frac{\sqrt{2}}{2}} = \frac{1}{2} + \log\frac{\sqrt{2}}{2} - \log 1 = \frac{1}{2} + \log\frac{\sqrt{2}}{2}.$

3. 積を和に直す公式より $\cos mx\cos nx = \dfrac{1}{2}\{\cos(mx+nx)+\cos(mx-nx)\} = \dfrac{1}{2}\{\cos(m+n)x+\cos(m-n)x\}$

$m\neq n$ のとき与式 $\displaystyle= \frac{1}{2}\int_{-\pi}^{\pi}\{\cos(m+n)x+\cos(m-n)x\}dt = \frac{1}{2}\left[\frac{1}{m+n}\sin(m+n)x + \frac{1}{m-n}\sin(m-n)x\right]_{-\pi}^{\pi}$

$= \dfrac{1}{2(m+n)}\sin(m+n)\pi + \dfrac{1}{2(m-n)}\sin(m-n)\pi - \dfrac{1}{2(m+n)}\sin\{-(m+n)\pi\} - \dfrac{1}{2(m-n)}\sin\{-(m-n)\pi\} = 0.$

$m=n$ のとき $\cos(m-n)x = \cos 0 = 1$ だから第 2 項の積分が $\dfrac{1}{2}\displaystyle\int_{-\pi}^{\pi} dx = \dfrac{1}{2}[x]_{-\pi}^{\pi} = \dfrac{1}{2}\{\pi-(-\pi)\} = \pi.$

4. 公式 $\tan^2 x + 1 = \dfrac{1}{\cos^2 x}$ より $\tan^2 x = \dfrac{1}{\cos^2 x} - 1.$ よって

$I_n = \displaystyle\int \tan^{n-2} x \cdot \tan^2 x dx = \int \tan^{n-2} x \left(\dfrac{1}{\cos^2 x} - 1\right) dx = \int \tan^{n-2} x \dfrac{1}{\cos^2 x} dx - \int \tan^{n-2} x dx.$

第 1 項の積分は $\tan x = t$ とおくと $\dfrac{1}{\cos^2 x}dx = dt.$ 第 2 項の積分は I_{n-2} だから

$I_n = \int t^{n-2}dt - I_{n-2} = \dfrac{t^{n-1}}{n-1} - I_{n-2} = \dfrac{1}{n-1}\tan^{n-1} x - I_{n-2}.$

5. 左辺の積分において $x = \dfrac{1}{t}$ とおくと $dx = -\dfrac{1}{t^2}dt, \dfrac{1}{x} = t,$

x	1	$\to$	2
t	1	$\to$	$\frac{1}{2}$

だから

左辺 $= \displaystyle\int_1^{\frac{1}{2}} \dfrac{f\left(\frac{1}{t}\right)}{\frac{1}{t}} \left(-\dfrac{1}{t^2}dt\right) = -\int_1^{\frac{1}{2}} \dfrac{f\left(\frac{1}{t}\right)}{t}dt = \int_{\frac{1}{2}}^1 \dfrac{f\left(\frac{1}{t}\right)}{t}dt$

題意より $f\left(\dfrac{1}{t}\right) = f(t)$ だから 左辺 $= \displaystyle\int_{\frac{1}{2}}^1 \dfrac{f(t)}{t}dt =$ 右辺.