

P. 56

第 6 章 偏微分法

§1. 偏微分

$$1.1 \quad (1) \quad z = (x^2 + y)^3 \quad z_x = 6x(x^2 + y)^2 \quad z_y = 3(x^2 + y)^2$$

$$(2) \quad z = \log(\sin x + \cos y) \quad z_x = \frac{\cos x}{\sin x + \cos y} \quad z_y = \frac{-\sin y}{\sin x + \cos y}$$

$$(3) \quad z = \sin^{-1} \frac{xy}{\sqrt{x^2 + y^2}} \quad z_x = \frac{1}{\sqrt{1 - \frac{y^2}{x^2 + y^2}}} \cdot \frac{xy}{-2(\sqrt{x^2 + y^2})^3} = \frac{-y \sin^{-1}(x)}{x^2 + y^2}$$

$$\left(\operatorname{sign}(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases} \right) \quad z_y = \frac{1}{\sqrt{1 - \frac{y^2}{x^2 + y^2}}} \left(\frac{1}{\sqrt{x^2 + y^2}} - \frac{y^2}{\sqrt{x^2 + y^2}} \right) = \frac{1 \cdot x}{x^2 + y^2}$$

$$(4) \quad z = \frac{xy}{\sqrt{x^2 + y^2}} \quad z_x = \frac{y}{\sqrt{x^2 + y^2}} - \frac{y \cdot y^2}{(\sqrt{x^2 + y^2})^3} = \frac{y^3}{(\sqrt{x^2 + y^2})^3}$$

$$z_y = \frac{x}{\sqrt{x^2 + y^2}} - \frac{x \cdot x^2}{(\sqrt{x^2 + y^2})^3} = \frac{y^3}{(\sqrt{x^2 + y^2})^3}$$

$$(5) \quad z = \frac{y}{x^2 + y^2 + 1} \quad z_x = \frac{-2xy}{(x^2 + y^2 + 1)^2} \quad z_y = \frac{x^2 + y^2 + 1 - 2y^2}{(x^2 + y^2 + 1)^2} = \frac{x^2 - y^2 + 1}{(x^2 + y^2 + 1)^2}$$

$$(6) \quad z = \sin(x \cos y - y \sin x) \quad z_x = \cos(x \cos y - y \sin x) (\cos y - y \cos x)$$

$$z_y = \cos(x \cos y - y \sin x) (-x \sin y - \sin x)$$

$$(7) \quad z = \tan^{-1} \frac{y}{2x} \quad z_x = \frac{1}{1 + \left(\frac{y}{2x}\right)^2} \cdot \frac{-y}{2x^2} = \frac{-2y}{4x^2 + y^2}$$

$$z_y = \frac{1}{1 + \left(\frac{y}{2x}\right)^2} \cdot \frac{1}{2x} = \frac{2x}{4x^2 + y^2}$$

$$1.2 \quad (1) \quad z = \tan^{-1} \frac{y}{x} \quad z_x = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2} \quad z_{xx} = \frac{2xy}{(x^2 + y^2)^2}$$

$$z_y = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2} \quad z_{yy} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\therefore z_{xx} + z_{yy} = 0$$

$$(2) \quad z = \log(x^2 + y^2) \quad z_x = \frac{2x}{x^2 + y^2} \quad z_{xx} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}$$

$$z_y = \frac{2y}{x^2 + y^2} \quad z_{yy} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}$$

$$\therefore z_{xx} + z_{yy} = 0$$

$$(3) \quad z = \log \sqrt{x^2 + y^2} = \frac{1}{2} \log(x^2 + y^2) \quad z_x = \frac{x}{x^2 + y^2} \quad z_{xx} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$z_y = \frac{y}{x^2 + y^2} \quad z_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\therefore z_{xx} + z_{yy} = 0$$

P. 56

$$1.2 \quad (5) \quad z = x^3y + 2x^2y^2 + 5xy^3, \quad z_x = 3x^2y + 4xy^2 + 5y^3, \quad z_{xx} = 6xy + 4y^2$$

$$z_y = x^3 + 4x^2y + 15xy^2, \quad z_{yy} = 4x^2 + 30xy$$

$$\therefore z_{xx} + z_{yy} = 4(x^2y^2 + 9xy)$$

$$(6) \quad z = \tan^{-1} \frac{x}{y} \quad z_x = \frac{y}{x^2+y^2} \quad z_{xx} = \frac{-2xy}{(x^2+y^2)^2}$$

$$z_y = \frac{-x}{x^2+y^2} \quad z_{yy} = \frac{2xy}{(x^2+y^2)^2}$$

$$\therefore z_{xx} + z_{yy} = 0$$

$$(7) \quad z = e^{ax} (\sin by + \cos by) \quad z_{xx} = a^2 e^{ax} (\sin by + \cos by)$$

$$z_{yy} = -b^2 e^{ax} (\sin by + \cos by)$$

$$z_{xx} + z_{yy} = (a^2 - b^2) z$$

$$1.3 \quad f = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$f_x = \frac{-x}{(\sqrt{x^2+y^2+z^2})^3} \quad f_{xx} = \frac{3x^2 - (x^2+y^2+z^2)}{(\sqrt{x^2+y^2+z^2})^5}$$

$$f_y = \frac{-y}{(\sqrt{x^2+y^2+z^2})^3} \quad f_{yy} = \frac{3y^2 - (x^2+y^2+z^2)}{(\sqrt{x^2+y^2+z^2})^5}$$

$$f_z = \frac{-z}{(\sqrt{x^2+y^2+z^2})^3} \quad f_{zz} = \frac{3z^2 - (x^2+y^2+z^2)}{(\sqrt{x^2+y^2+z^2})^5}$$

$$\therefore f_{xx} + f_{yy} + f_{zz} = 0$$

$$1.4 \quad z = e^x f(x+y) + e^{-x} g(x-y) \quad \frac{d}{dx} f(x) = f'(x), \quad \frac{d}{dx} g(x) = g'(x), \text{ etc.}$$

$$z_x = e^x \{f(x+y) + f'(x+y)\} + e^{-x} \{-g(x-y) + g'(x-y)\}$$

$$z_{xx} = e^x \{f(x+y) + 2f'(x+y) + f''(x+y)\} + e^{-x} \{g(x-y) - 2g'(x-y) + g''(x-y)\}$$

$$z_{yy} = e^x f''(x+y) + e^{-x} g''(x-y)$$

$$\therefore z_{xx} = z_{yy} + 2[e^x f'(x+y) - e^{-x} g'(x-y)] + e^x f(x+y) + e^{-x} g(x-y)$$

$$= z_{yy} + 2z_x + z$$

$$1.5 \quad z = f(ax+by) \quad z_x = a f'(ax+by) \quad z_y = b f'(ax+by)$$

$$\therefore b z_x - a z_y = 0$$

$$1.6 \quad z = f(x, y) \quad z_{xx} + z_{yy} = 0 \quad \text{or } z \text{ is } u = y z_x - x z_y$$

$$u_x = y z_{xx} - z_y - x z_{yx} \quad u_{xx} = y z_{xxx} - 2z_{xy} - x z_{yxx}$$

$$u_y = z_x + y z_{xy} - x z_{yy} \quad z_{yy} = z z_{xy} + y z_{xyy} - x z_{yyx}$$

P.56

$$\begin{aligned} \therefore u_{zx} + u_{yz} &= y(z_{xx} + z_{yy}) - x(z_{yx} + z_{xy}) \\ &= y \frac{\partial}{\partial x} (z_{xx} + z_{yy}) - x \frac{\partial}{\partial y} (z_{xx} + z_{yy}) = 0 \end{aligned}$$

1.7 $z = \cos(xy)$

$$\begin{aligned} z_x &= -\sin(xy) \cdot y & z_{xy} &= -\cos(xy) \cdot xy - \sin(xy) \\ & & &= -(xy \cos(xy) + \sin(xy)) \end{aligned}$$

1.8 $u = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} \quad u_x = \begin{vmatrix} 0 & 1 & 1 \\ 1 & y & z \\ 2x & y^2 & z^2 \end{vmatrix} \quad u_{xx} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & y & z \\ 2 & y^2 & z^2 \end{vmatrix}$

同样求 $u_{yy} = \begin{vmatrix} 1 & 0 & 1 \\ x & 0 & z \\ x^2 & 2 & z^2 \end{vmatrix} \quad u_{zz} = \begin{vmatrix} 1 & 1 & 0 \\ x & y & 0 \\ x^2 & y^2 & 2 \end{vmatrix}$

$$\therefore u_{xx} + u_{yy} + u_{zz} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 2 & 2 & 2 \end{vmatrix} = 0$$

1.9 $x + y + z = f(x^2 + y^2 + z^2) \quad \frac{d}{dt} f(x) = f'(x) \cdot \dot{x} < \dot{x} <$

(1) $1 + z_x = 2x f'(x^2 + y^2 + z^2) + 2z z_x f'(x^2 + y^2 + z^2)$

$$z_x (1 - 2z f'(x^2 + y^2 + z^2)) = 2x f'(x^2 + y^2 + z^2) - 1$$

$$z_x = \frac{2x f'(x^2 + y^2 + z^2) - 1}{1 - 2z f'(x^2 + y^2 + z^2)}$$

(2) $z_y = \frac{2y f'(x^2 + y^2 + z^2) - 1}{1 - 2z f'(x^2 + y^2 + z^2)}$

$$(y-z) z_x + (z-x) z_y = \frac{2x(y-z) f'(x^2 + y^2 + z^2) - (y-z) + 2y(z-x) f'(x^2 + y^2 + z^2) - (z-x)}{1 - 2z f'(x^2 + y^2 + z^2)}$$

$$= \frac{2z(y-x) f'(x^2 + y^2 + z^2) - (y-x)}{1 - 2z f'(x^2 + y^2 + z^2)} = -(y-x)$$

$$\therefore (y-z) z_x + (z-x) z_y = x-y$$

1.10 $z = f(r) \quad r = \sqrt{x^2 + y^2}$

$$z_x = f'(r) \frac{x}{\sqrt{x^2 + y^2}} \quad z_{xx} = f''(r) \frac{x^2}{x^2 + y^2} + f'(r) \left(\frac{1}{\sqrt{x^2 + y^2}} - \frac{x^2}{(x^2 + y^2)^{3/2}} \right)$$

$$= f''(r) \frac{x^2}{r^2} + f'(r) \frac{y^2}{r^3}$$

$$z_{xy} = f''(r) \frac{xy}{r^2} - f'(r) \frac{xy}{(x^2 + y^2)^{3/2}}$$

$$= f''(r) \frac{xy}{r^2} - f'(r) \frac{xy}{r^3}$$

$$z_{yy} = f''(r) \frac{y^2}{r^2} + f'(r) \frac{x^2}{r^3}$$

P. 56

1.11 1.9 1: 174 U

$$1.12 \quad f(x, y) = x^3 a^{\sin y} \quad f_x = 3x^2 a^{\sin y} \quad f_y = x^3 \log a \cdot e^{\sin y} \cos y \\ = x^3 a^{\sin y} (\log a \cdot \cos y) \\ f_{xy} = 3x^2 \log a \cdot a^{\sin y} \cos y \\ = 3x^2 a^{\sin y} (\log a \cdot \cos y)$$

1.13 $u = \log(x^2 + y^2 + z^2 - 3xyz)$

(1) $u_x = \frac{3(x^2 - yz)}{x^2 + y^2 + z^2 - 3xyz}$

$$(2) \quad u_x + u_y + u_z = \frac{3(x^2 - yz + y^2 - xz + z^2 - xy)}{x^2 + y^2 + z^2 - 3xyz} \\ = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x+y)^2 + z^2 - 3xy(x+y+z)} \\ = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x+y+z)[(x+y)^2 + (x+y)z + z^2 - 3xy]} \\ = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)} \\ = \frac{3}{x+y+z}$$

P. 57

§ 2 合成関数の偏微分

$$2.1 (1) z = \tan^{-1} \frac{y}{x}, \quad x = 1 + \sin t, \quad y = 1 - \cos t$$

$$\begin{aligned} \frac{dz}{dt} &= \frac{1}{1 + \frac{y^2}{x^2}} \left(-\frac{y}{x^2}\right) \cdot (1 + \cos t) + \frac{1}{1 + \frac{y^2}{x^2}} \frac{1}{x} \sin t \\ &= \frac{-\frac{y}{x^2}(1 + \cos t) + \frac{1}{x} \sin t}{x^2 + y^2} \end{aligned}$$

$$(2) z = f(x, y) \quad x = g(t) \quad y = h(t)$$

$$\frac{dz}{dt} = f_x(x, y) g'(t) + f_y(x, y) h'(t)$$

$$2.2 (1) z = x^2 + y^2, \quad x = 2u - v, \quad y = u + 2v$$

$$z_u = 2x \cdot 2 + 2y \cdot 1 = 4x + 2y = 10u$$

$$z_v = 2x \cdot (-1) + 2y \cdot 2 = 2(-x + 2y) = 10v$$

$$(2) z = xy, \quad x = \log \sqrt{u^2 + v^2} = \frac{1}{2} \log(u^2 + v^2) \quad y = \tan^{-1} \frac{u}{v}$$

$$\begin{aligned} z_u &= y x_u + x y_u = y \frac{u}{u^2 + v^2} + x \frac{1}{1 + \frac{u^2}{v^2}} \frac{1}{v} \\ &= \frac{y u + x v}{u^2 + v^2} = \frac{1}{u^2 + v^2} \left(u \tan^{-1} \frac{u}{v} + v \log \sqrt{u^2 + v^2} \right) \end{aligned}$$

$$\begin{aligned} z_v &= y x_v + x y_v = y \frac{v}{u^2 + v^2} + x \frac{-u}{u^2 + v^2} \\ &= \frac{1}{u^2 + v^2} \left(v \tan^{-1} \frac{u}{v} - u \log \sqrt{u^2 + v^2} \right) \end{aligned}$$

$$2.3 \quad z = f(x, y) \quad x = u + v, \quad y = uv$$

$$z_u = z_x + z_y v$$

$$z_{uv} = z_{xx} + z_{xy} u + z_{yx} v + z_{yy} uv + z_y$$

$$= z_{xx} + (u+v) z_{xy} + z_{yy} uv + z_y$$

$$= z_{xx} + 2z_{xy} + z_{yy} \cdot uv$$

$$2.4 \quad z = f(x, y) \quad x = r \cos \theta \quad y = r \sin \theta$$

$$z_r = z_x \cos \theta + z_y \sin \theta$$

$$z_{rr} = z_{xx} \cos^2 \theta + 2z_{xy} \cos \theta \sin \theta + z_{yy} \sin^2 \theta$$

$$z_\theta = z_x (-r \sin \theta) + z_y r \cos \theta$$

$$z_{\theta\theta} = z_{xx} r^2 \sin^2 \theta - 2z_{xy} r^2 \sin \theta \cos \theta + z_{yy} r^2 \cos^2 \theta - z_x r \cos \theta - z_y r \sin \theta$$

$$\therefore z_{rr} + \frac{1}{r^2} z_{\theta\theta} = z_{xx} + z_{yy} - \frac{1}{r} z_r$$

$$\therefore z_{xx} + z_{yy} = z_{rr} + \frac{1}{r^2} z_{\theta\theta} + \frac{1}{r} z_r$$

P.57

$$2.5 \quad z = f(x, y) \quad x = r \cos \theta \quad y = r \sin \theta$$

$$z_r = z_x \cos \theta + z_y \sin \theta$$

$$z_\theta = -r z_x \sin \theta + r z_y \cos \theta$$

$$\therefore (z_r)^2 + \frac{1}{r^2} (z_\theta)^2 = (z_x \cos \theta + z_y \sin \theta)^2 + (-z_x \sin \theta + z_y \cos \theta)^2 \\ = (z_x)^2 + (z_y)^2$$

$$2.6 \quad u = 2x + 3y, \quad v = 4x - 5y, \quad z = f(u, v)$$

$$\begin{cases} 5u + 3v = 22x \\ 2u - v = 11y \end{cases} \quad \begin{cases} x = \frac{1}{22} (5u + 3v) \\ y = \frac{1}{11} (2u - v) \end{cases}$$

$$z_u = z_x \frac{5}{22} + z_y \frac{2}{11} = \frac{1}{22} (5z_x + 4z_y)$$

$$z_v = z_x \frac{3}{22} + z_y \frac{-1}{11} = \frac{1}{22} (3z_x - 2z_y)$$

$$2.7 \quad z = f(x, y) \quad x = \rho \cos \alpha - \rho \sin \alpha, \quad y = \rho \sin \alpha + \rho \cos \alpha \quad (\text{向題集 } \Sigma 27^\circ) \quad \text{f}$$

$$z_\rho = z_x \cos \alpha + z_y \sin \alpha \quad z_{\rho\rho} = z_{xx} \cos^2 \alpha + 2z_{xy} \cos \alpha \sin \alpha + z_{yy} \sin^2 \alpha$$

$$z_\alpha = z_x (-\sin \alpha) + z_y \cos \alpha \quad z_{\alpha\alpha} = z_{xx} \sin^2 \alpha - 2z_{xy} \cos \alpha \sin \alpha + z_{yy} \cos^2 \alpha$$

$$\therefore z_{\rho\rho} + z_{\alpha\alpha} = z_{xx} + z_{yy}$$

$$2.8 \quad z = f(r) \quad r = \sqrt{x^2 + 2y^2}$$

$$z_x = f'(r) \frac{x}{\sqrt{x^2 + 2y^2}} \quad z_{xx} = f''(r) \frac{x^2}{x^2 + 2y^2} + f'(r) \frac{1}{\sqrt{x^2 + 2y^2}} - f'(r) \frac{x^2}{(\sqrt{x^2 + 2y^2})^3}$$

$$= f''(r) \frac{x^2}{r^2} + f'(r) \frac{2y^2}{r^3}$$

$$z_y = f'(r) \frac{2y}{\sqrt{x^2 + 2y^2}} \quad z_{yy} = f''(r) \frac{4y^2}{r^2} + f'(r) \frac{2}{\sqrt{x^2 + 2y^2}} - f'(r) \frac{4y^2}{(\sqrt{x^2 + 2y^2})^3}$$

$$= f''(r) \frac{4y^2}{r^2} + f'(r) \frac{2x^2}{r^3}$$

$$\therefore z_{xx} + \frac{1}{2} z_{yy} = f''(r) \frac{x^2 + 2y^2}{r^2} + f'(r) \frac{2y^2 + x^2}{r^3} = f''(r) + f'(r) \frac{1}{r}$$

$$\therefore z_{xx} + \frac{1}{2} z_{yy} = f''(r) + \frac{f'(r)}{r}$$

§.3 連続と偏微分

P 57

$$3.1 \quad f(x, y) = \begin{cases} x + y + \frac{xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$(1) \quad f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{h - 0}{h} = 1$$

$$(2) \quad y = x \pm \varepsilon \quad x \rightarrow 0 \text{ かつ } \varepsilon$$

$$\lim_{x \rightarrow 0} f(x, y) = \lim_{x \rightarrow 0} (2x + \frac{x^2}{2x^2}) = \frac{1}{2} \neq f(0, 0)$$

= 原点で不連続

$$3.2 \quad f(x, y) = \frac{\log |ax^2 + by^2 - 1|}{x^2 + y^2}$$

$$(1) \quad \lim_{y \rightarrow 0} \left\{ \lim_{x \rightarrow 0} \frac{\log |ax^2 + by^2 - 1|}{x^2 + y^2} \right\} = \lim_{y \rightarrow 0} \frac{\log |by^2 - 1|}{y^2}$$

$$= \lim_{y \rightarrow 0} \frac{\frac{2by}{by^2 - 1}}{2y} = \lim_{y \rightarrow 0} \frac{b}{by^2 - 1} = -b$$

(2) (1) と同様

$$\lim_{x \rightarrow 0} \left\{ \lim_{y \rightarrow 0} \frac{\log |ax^2 + by^2 - 1|}{x^2 + y^2} \right\} = -a$$

$$\therefore a = b$$

$$3.3 \quad \Delta u = \lim_{h \rightarrow 0} \frac{u(x-h, y) + u(x+h, y) + u(x, y-h) + u(x, y+h) - 4u(x, y)}{h^2}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h^2} \{ u(x-h, y) - u(x, y) + u(x, y+h) - u(x, y) + u(x+h, y) - u(x, y) + u(x, y-h) - u(x, y) \}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h^2} \left\{ -h u_x(x, y) + \frac{h^2}{2} u_{xx}(x, y) + h u_y(x, y) + \frac{h^2}{2} u_{yy}(x, y) + h u_x(x, y) + \frac{h^2}{2} u_{xx}(x, y) - h u_y(x, y) + \frac{h^2}{2} u_{yy}(x, y) \right\}$$

$$= u_{xx} + u_{yy}$$

$$3.4 \quad f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$\lim_{x=y \rightarrow 0} f(x, y) = \lim_{x \rightarrow 0} \frac{x^2}{2x^2} = \frac{1}{2} \neq f(0, 0) \quad \therefore (0, 0) \text{ で不連続}$$

$$f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = 0$$

$$f_y(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = 0$$

P 57

3.5

$$f(x, y) = \begin{cases} \frac{xy(e^{x^2} - e^{y^2})}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$(1) f_x(x, y) = \frac{y(e^{x^2} - e^{y^2})(2x + y^2) + 2x^2y(x^2 + y^2)e^{x^2} - 2x^2y(e^{x^2} - e^{y^2})}{(x^2 + y^2)^2}$$

$$= \frac{y(e^{x^2} - e^{y^2})(y^2 + x^2) + 2x^2y(x^2 + y^2)e^{x^2}}{(x^2 + y^2)^2}$$

$$(2) f_y(x, y) = \frac{(x(e^{x^2} - e^{y^2}) - 2xy^2e^{y^2})(x^2 + y^2) - 2xy^2(e^{x^2} - e^{y^2})}{(x^2 + y^2)^2}$$

$$= \frac{x(x^2 + y^2)(e^{x^2} - e^{y^2}) - 2xy^2(e^{x^2} - e^{y^2})}{(x^2 + y^2)^2}$$

$$(3) f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = 0$$

$$f_y(0, 0) = \lim_{k \rightarrow 0} \frac{f(0, k) - f(0, 0)}{k} = 0$$

$$f_{xy}(0, 0) = \lim_{k \rightarrow 0} \frac{f_x(0, k) - f_x(0, 0)}{k} = \lim_{k \rightarrow 0} \frac{k(1 - e^{k^2})k^2}{k^3} = -1 \quad *$$

$$f_{yx}(0, 0) = \lim_{h \rightarrow 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{h^3(e^{h^2} - 1)}{h^3}$$

$$= \lim_{h \rightarrow 0} \frac{1 + h^2 + o(h^2) - 1}{h^2} = 1$$

$$* e^{h^2} = 1 + h^2 + \frac{h^4}{2} + \frac{h^6}{3!} + \dots$$

$$= 1 + h^2 + o(h^2)$$

$o(h^2)$ 4位の無限小

P.58

§4 偏微分の応用1 (最大・最小)

4.1(1) $z = x^4 - x^3 - y^3 + 3x^2y$

$$z_x = 4x^3 - 3x^2 + 6xy = x(4x^2 - 3x + 6y)$$

$$z_y = -3y^2 + 3x^2 = 3(x-y)(x+y)$$

$$z_{xx} = 12x^2 - 6x + 6y$$

$$z_{xy} = 6x$$

$$z_{yy} = -6y$$

$$\frac{1}{36}(z_{xx}z_{yy} - z_{xy}^2) = -\frac{1}{6}(2x^2 - x + y) - x^2$$

$$(0, 0) \text{ のときは } 0$$

$$\left(-\frac{3}{4}, -\frac{3}{4}\right) \text{ のときは } \frac{3}{4} \cdot \frac{9}{8} - \frac{9}{16} > 0 \quad f_{xx} > 0$$

$$\left(-\frac{3}{4}, -\frac{3}{4}\right) \text{ のときは 極小値 } \frac{81}{4^4} + \frac{27}{4^3} + \frac{27}{4^3} - \frac{81}{4^3} = \frac{-27}{4^3}$$

$$\left(\frac{9}{4}, -\frac{9}{4}\right) \text{ のときは } \frac{9}{4} \left(\frac{81}{8} - \frac{9}{4} - \frac{9}{4}\right) - \frac{81}{16} = \frac{45 - 162}{32} > 0 \quad f_{xx} > 0$$

$$\left(\frac{9}{4}, -\frac{9}{4}\right) \text{ のときは 極小値 } \left(\frac{9}{4}\right)^4 - \left(\frac{9}{4}\right)^3 + \left(\frac{9}{4}\right)^3 - 3\left(\frac{9}{4}\right)^2 = \frac{-3^7}{4^3}$$

$$x(4x^2 - 3x + 6y) = 0$$

$$y = \frac{1}{2}x$$

$$y = x \text{ かつ}$$

$$x^2(4x + 3) = 0$$

$$y = -x$$

$$x^2(4x - 9) = 0$$

$$\therefore (0, 0), \left(-\frac{3}{4}, -\frac{3}{4}\right), \left(\frac{9}{4}, -\frac{9}{4}\right)$$

2) $z = x^3 - 2x^2y + x^2 - y^2$

$$z_x = 3x^2 - 4xy + 2x = x(3x - 4y + 2)$$

$$z_y = -2x^2 - 2y = -2(y + x^2)$$

$$(0, 0)$$

$$z_{xx} = 6x - 4y + 2$$

$$z_{xy} = -4x$$

$$z_{yy} = -2$$

$$z_{xx}z_{yy} - z_{xy}^2 = -2(6x - 4y + 2) - 16x^2$$

$$(0, 0) \text{ のときは}$$

∴ 極値なし

$$x(3x - 4y + 2) = 0$$

$$y + x^2 = 0$$

$$x = 0$$

$$3x + 4x^2 + 2 = 0$$

$$\text{解なし}$$

3) $z = x^3 + 2x^2y - xy^2 - 4xy$

$$z_x = 3x^2 + 4xy - y^2 - 4y$$

$$z_y = 2x^2 - 2xy - 4x = 2x(x - y - 2)$$

$$\therefore (0, 0), (0, -4)$$

$$z_{xx} = 6x + 4y$$

$$z_{xy} = 4x - 2y - 4$$

$$z_{yy} = -2x$$

∴ 極値なし

$$\begin{cases} 3x^2 + 4xy - y^2 - 4y = 0 \\ x(x - y - 2) = 0 \end{cases}$$

$$x = 0$$

$$y = x - 2$$

$$y = 0$$

$$3x^2 + 4x^2 - 8x - x^2 - 4x - 4 - 4x + 8$$

$$y = -4$$

$$6x^2 - 8x + 4 = 0$$

$$\text{解なし}$$

$$z_{xx}z_{yy} - z_{xy}^2 = -4(3x + 2y)x - 4(2x - y - 1)^2$$

$$(0, 0) \text{ のときは } < 0, (0, -4) \text{ のときは } < 0$$

P. 58

4.1 (4) $Z = x^4 + y^4 - 2x^2 - 2y^2 = 4xy$

$Z_x = 4x^3 - 4x - 4y$

$Z_y = 4y^3 - 4y - 4x$

$(0, 0) (\sqrt{2}, \sqrt{2}) (-\sqrt{2}, -\sqrt{2})$

$Z_{xx} = 12x^2 - 4$

$Z_{xy} = -4$

$Z_{yy} = 12y^2 - 4$

$D = Z_{xx}Z_{yy} - Z_{xy}^2 = 16(3x^2 - 1)(3y^2 - 1) - 16$

$(0, 0)$ のとき $D = 0$ $(\pm\sqrt{2}, \pm\sqrt{2})$ のときは $D > 0$

$(0, 0)$ に近い点, $z = z > 0, z < 0$ となるので 極値を とれない

$(\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})$ のとき $f_{xx} > 0$: 極小値 $4 + 4 - 4 - 4 = -8$ とれる

(5) $Z = (x^2 + y^2)^2 - 2(x^2 + y^2)$

$Z_x = 4x(x^2 + y^2) - 4x = 4x(x^2 + y^2 - 1) = 0$ $(0, 0) (1, 0) (-1, 0)$

$Z_y = 4y(x^2 + y^2) + 4y = 4y(x^2 + y^2 + 1) = 0$

$Z_{xx} = 12x^2 + 4y^2 - 4$

$Z_{xy} = 8xy$

$Z_{yy} = 12y^2 + 4x^2 + 4$

$D = 16(3x^2 + y^2 - 1)(3y^2 + x^2 + 1) - 16 \cdot 8xy$
 $= 16\{(3x^2 + y^2 - 1)(3y^2 + x^2 + 1) - 4xy\}$

$(0, 0)$ のときは $D < 0$ $f_{xx} < 0$: 極値なし

$(1, 0)$ のときは $D > 0$ $f_{xx} > 0$: 極小値 -1

$(-1, 0)$ のときは $D > 0$ $f_{xx} > 0$: 極小値 -1

(6) $Z = x^3 + 3xy^2 + y^3$

$Z_x = 3x^2 + 3y$ $x^2 + y = 0$ $y = -x^2$ $x(1 + x^2) = 0$

$Z_y = 3x + 3y^2$ $x + y^2 = 0$ $(0, 0) (-1, -1)$

$Z_{xx} = 6x$

$Z_{yy} = 6y$ $D = 9(4xy - 1)$

$Z_{xy} = 3$ $(0, 0)$ のとき $D < 0$ 極値なし $(-1, -1)$ のとき $D > 0$ $Z_{xx} < 0$

$(-1, -1)$ のとき 極大値 1

(7) $Z = xy - x^2y - xy^3$

$Z_x = y - 2x^2y - y^3 = y(1 - 2x^2 - y^2)$

$Z_y = x - x^2 - 3xy^2 = x(1 - 3y^2 - x^2)$

$y(1 - 2x^2 - y^2) = 0$

$x(1 - 3y^2 - x^2) = 0$

$(0, 0) (\pm\frac{1}{2}, \pm\frac{1}{2}) (\pm 1, 0) (0, \pm 1)$

P.58

$$z_{xx} = -6xy$$

$$z_{xy} = 1 - 3x^2 - 3y^2$$

$$z_{yy} = -6xy$$

$$D = 36x^2y^2 - (1 - 3x^2 - 3y^2)^2$$

$$(0, 0) \text{ 附近 } D < 0 \text{ 極大極小}$$

$$(\pm 1, 0) \text{ 附近 } D < 0 \text{ "}$$

$$(0, \pm 1) \text{ " " "}$$

$$(\pm \frac{1}{2}, \pm \frac{1}{2}) \text{ 附近 } D = \frac{9}{4} - (1 - \frac{3}{2})^2 > 0$$

$$(\frac{1}{2}, \frac{1}{2}) (-\frac{1}{2}, -\frac{1}{2}) \text{ 附近 } z_{xx} < 0 \text{ 極大值 } \frac{1}{8}$$

$$(\frac{1}{2}, -\frac{1}{2}) (-\frac{1}{2}, \frac{1}{2}) \text{ 附近 } z_{xx} > 0 \text{ 極小值 } -\frac{1}{8}$$

$$(8) \quad z = x^3 + 8y^3 + 12axy \quad (a \neq 0)$$

$$z_x = 3x^2 + 12ay = 3(x^2 + 4ay)$$

$$x^2 + 4ay = 0 \quad y = -\frac{x^2}{4a}$$

$$2y^2 + ax = 0$$

$$\frac{x^4}{8a^2} + ax = 0$$

$$z_y = 24y^2 + 12ax = 12(2y^2 + ax)$$

$$x(x^2 + 8a^3) = 0$$

$$(0, 0) \quad (-2a, -a)$$

$$\left\{ \begin{array}{l} z_{xx} = 6x \\ z_{yy} = 48y \\ z_{xy} = 12a \end{array} \right.$$

$$D = 12^2(2xy - a^2)$$

$$(0, 0) \text{ 附近 } D < 0 \text{ 極大極小}$$

$$(-2a, -a) \text{ 附近 } D > 0$$

$$a > 0 \text{ 附近 } z_{xx} = -6a < 0 \text{ 極大值 } 8a^3$$

$$a < 0 \text{ 附近 } z_{xx} = -6a > 0 \text{ 極小值 } 8a^3$$

$$(9) \quad z = xy(x + 2y - 6) = x^2y + 2xy^2 - 6xy$$

$$z_x = 2xy + 2y^2 - 6y = (2x + 2y - 6)y$$

$$(x + y - 3)y = 0$$

$$(x + 4y - 6)x = 0$$

$$z_y = x^2 + 4xy - 6x = (x + 4y - 6)x$$

$$\begin{array}{l} x=0 \quad y=0 \quad y=1 \\ y=0,3 \quad x=6 \quad x=1 \end{array}$$

$$(0, 0), (0, 3), (6, 0), (2, 1)$$

$$z_{xx} = 2y \quad z_{xy} = 2x + 4y - 6 \quad z_{yy} = 4x$$

$$D = z_{xx}z_{yy} - (z_{xy})^2 = 8xy - 4(x + 2y - 3)^2 = 4(2xy - (x + 2y - 3)^2)$$

$$(0, 0) \text{ 附近 } D < 0, (0, 3) \text{ 附近 } D < 0, (6, 0) \text{ 附近 } D < 0, (2, 1) \text{ 附近 } D > 0$$

$$(2, 1) \text{ 附近 } z_{xx} > 0 \quad \therefore (2, 1) \text{ 附近 } \text{極小值 } -4$$

P. 58

(10) $z = x^3 + 2x^2y + xy^2 - 4xy$

$z_x = 3x^2 + 4xy + y^2 - 4y$

$z_y = 2x^2 + 2xy - 4x$

$(0, 0) \quad (0, 4) \quad \left(\frac{1}{2}, \frac{3}{2}\right)$

$z_{xx} = 6x + 4y$

$z_{xy} = 4x + 2y - 4$

$z_{yy} = 2x$

$D = 4(3x + 2y)x - 4(2x + y - 2)^2$

$= 4\{(3x + 2y)x - (2x + y - 2)^2\}$

$(0, 0) z'' D < 0 \quad (0, 4) z'' D < 0 \quad \left(\frac{1}{2}, \frac{3}{2}\right) z'' \frac{D}{4} = \frac{9}{4} - \frac{1}{4} > 0$

$z_{xx} = \frac{6}{2} + \frac{12}{2} > 0$

 $\left(\frac{1}{2}, \frac{3}{2}\right) z''$ 極小值 -1 .

(11) $z = 4x^3 - y^3 + 3x^2y + 9y$

$z_x = 12x^2 + 6xy$

$z_y = -3y^2 + 3x^2 + 9$

$6x(2x + y) = 0$

$x = 0 \quad y = -2x$

$-y^2 + x^2 + 3 = 0$

$-3x^2 + 3 = 0$

$(0, \pm\sqrt{3})$

$(1, -2), (-1, 2)$

$z_{xx} = 24x + 6y$

$z_{xy} = 6x$

$z_{yy} = -6y$

$D = 36\{-y(4x + y) - x^2\}$

$(0, \pm\sqrt{3}) D < 0$

$(1, -2) D > 0$

$(-1, 2) D > 0$

$(1, -2) z_{xx} = 24 - 12 > 0 \quad (-1, 2) z_{xx} = -24 + 12 < 0$

$(1, -2) z''$ 極小值 $4 + 8 - 6 - 18 = -12$

$(-1, 2) z''$ 極大值 $-4 - 8 + 6 + 18 = 12$

(12) $z = x^2 - xy + y^2 - 3x + y - 2$

$z_x = 2x - y - 3$

$z_y = -x + 2y + 1$

$2x - y - 3 = 0$

$-2x + 2y + 2 = 0$

$3y - 1 = 0$

$y = \frac{1}{3} \quad x = \frac{5}{3}$

$z_{xx} = 2 \quad z_{xy} = -1 \quad z_{yy} = 2$

$D = z_{xx} z_{yy} - z_{xy}^2 = 4 - 1 > 0 \quad z_{xx} > 0$

$\left(\frac{5}{3}, \frac{1}{3}\right) z''$ 極小值 $\frac{25}{9} - \frac{1}{9} - \frac{1}{3} - \frac{5}{3} + \frac{1}{3} - 2 = -\frac{13}{9}$

P. 58

(13) $z = x^2 y^2 - 3(x-y)^2$

$$z_x = 4x^2 - 6(x-y)$$

$$z_y = 4y^2 + 6(x-y)$$

$$z_{xx} = 12x^2 - 6$$

$$z_{xy} = 6$$

$$z_{yy} = 12y^2 - 6$$

$$2x^2 - 3(x-y) = 0$$

$$2y^2 + 3(x-y) = 0$$

$$x^2 + y^2 = 0$$

$$y = -x$$

$$2(x^2 y^2) - 6(x-y) = 0$$

$$(x-y)(x^2 + y^2 - 3) = 0$$

$$y = x, x^2 + x^2 + y^2 = 3$$

$$(0,0) (\sqrt{3}, -\sqrt{3}) (-\sqrt{3}, \sqrt{3})$$

$$\frac{D}{36} = (2x^2 - 6)(2y^2 - 6) - 36$$

$(0,0)$ 处 $D=0$ $(0,0)$ 附近 $D < 0$ 正负不定 所以不是极值点

$(\pm\sqrt{3}, \mp\sqrt{3})$ 处 $D > 0$ $z_{xx} > 0$

$(\pm\sqrt{3}, \mp\sqrt{3})$ 处 极小值 $9 + 9 - 36 = -18$

(14) $z = x(1-x^2-y^2) = x - x^3 - xy^2$

$$z_x = 1 - 3x^2 - y^2$$

$$z_y = -2xy$$

$$z_{xx} = -6x$$

$$z_{xy} = -2y$$

$$z_{yy} = -2x$$

$$x = 0 \quad y = 0$$

$$y = \pm 1 \quad x = \pm \frac{1}{\sqrt{3}}$$

$$D = 12x^2 - 4y^2 \quad \frac{D}{4} = 3x^2 - y^2$$

$(0,0)$ 处 $D=0$ 极值点

$(0, \pm 1)$ 处 $D < 0$ 极值点 $(\pm \frac{1}{\sqrt{3}}, 0)$ 处 $D > 0$ $z_{xx} = -6(\pm \frac{1}{\sqrt{3}})$

$(\frac{1}{\sqrt{3}}, 0)$ 处 极大值 $\frac{2}{3\sqrt{3}}$ $(-\frac{1}{\sqrt{3}}, 0)$ 处 极小值 $-\frac{2}{3\sqrt{3}}$

* 2 (1) $z = (x^2 + 2y^2)e^{-(x^2 + y^2)}$

$$z_x = (2x - 2x(x^2 + 2y^2))e^{-(x^2 + y^2)}$$

$$z_y = (4y - 2y(x^2 + 2y^2))e^{-(x^2 + y^2)}$$

$$(0,0), (0, \pm 1), (\pm 1, 0)$$

$$z_{xx} = (2 - 6x^2 - 4y^2 - 4x^2 + 4x^2(x^2 + 2y^2))e^{-(x^2 + y^2)}$$

$$z_{yy} = (4 - 2x^2 - 12y^2 - 8y^2 + 4y^2(x^2 + 2y^2))e^{-(x^2 + y^2)}$$

$$z_{xy} = (-8xy - 4xy + 4xy(x^2 + 2y^2))e^{-(x^2 + y^2)}$$

$$D = z_{xx}z_{yy} - z_{xy}^2 \quad k < k < k$$

$(0,0)$ 处 $D > 0$ $z_{xx} > 0$ 极小值 0

$(0, \pm 1)$ 处 $D > 0$ $z_{xx} < 0$ 极大值 $2e^{-1}$

$(\pm 1, 0)$ 处 $D < 0$ 极值点

$$x(1-x^2-2y^2) = 0$$

$$y(2-x^2-2y^2) = 0$$

$$x = 0 \quad y = 0$$

$$y = \pm 1 \quad x = \pm 1$$

P.58

$$4.2(2) z = x e^{-(x^2+y^2)}$$

$$z_x = (1-2x^2) e^{-(x^2+y^2)} \quad \left(\pm \frac{1}{\sqrt{2}}, 0\right)$$

$$z_y = -2xy e^{-(x^2+y^2)}$$

$$z_{xx} = (-4x - 2x(1-2x^2)) e^{-(x^2+y^2)}$$

$$z_{xy} = -2y(1-2x^2) e^{-(x^2+y^2)}$$

$$z_{yy} = (-2x + 4xy^2) e^{-(x^2+y^2)} \quad D = z_{xx} z_{yy} - z_{xy}^2 \text{ etc.}$$

$$\left(\pm \frac{1}{\sqrt{2}}, 0\right) \text{ において } D > 0$$

$$\left(\frac{1}{\sqrt{2}}, 0\right) \text{ において } z_{xx} < 0 \quad \text{極大値} \quad \frac{1}{\sqrt{2}} e^{-\frac{1}{2}}$$

$$\left(-\frac{1}{\sqrt{2}}, 0\right) \text{ において } z_{xx} > 0 \quad \text{極小値} \quad -\frac{1}{\sqrt{2}} e^{-\frac{1}{2}}$$

$$(3) z = \sin x + \sin y + \sin(x+y)$$

$$\cos x - \cos y = 0 \quad 2 \sin \frac{x+y}{2} \sin \frac{x-y}{2} = 0$$

$$z_x = \cos x + \cos(x+y)$$

$$\frac{x+y}{2} = 0, \pi \quad \frac{x-y}{2} = 0$$

$$z_y = \cos y + \cos(x+y), \quad 0 < x, y < \pi$$

$$y = 2\pi - x \quad y = x$$

$$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$$

$$y = -x$$

$$\cos x + \cos 2x = 0$$

$$\cos x + 1 = 0$$

$$2 \cos^2 x + \cos x - 1 = 0$$

$$x = \pi$$

$$\cos x = \frac{1}{2}, -1$$

$$z_{xx} = -\sin x - \sin(x+y)$$

$$z_{yy} = -\sin y - \sin(x+y)$$

$$z_{xy} = -\sin(x+y)$$

$$D = z_{xx} z_{yy} - z_{xy}^2 > 0$$

$$\left(\frac{\pi}{3}, \frac{\pi}{3}\right) \text{ において } z_{xx} < 0 \quad \text{極大値} \quad \frac{3\sqrt{3}}{2}$$

$$4.3 \quad f(x, y) = x^2 - 3xy + y^2$$

$$(1) \quad f_x = 2x - 3y \quad f_y = 3y - 3x \quad f_{xx} = 2 \quad f_{yy} = -3 \quad f_{xy} = -3$$

$$(f_{xxy} = 0, f_{xyx} = f_{yxx} = 0, f_{yyx} = 0)$$

$$F(x, y) = f(x+h, y+k) - f(x, y)$$

$$= 3(x^2 - y)h + 3(y^2 - x)k + \frac{1}{2}(6xh^2 - 6hk^2 + 6yh^2)$$

$$= 3\left\{(x^2 - y)h + (y^2 - x)k + xh^2 - hk^2 + yk^2\right\}$$

$$(2) \quad (1, 1) \quad x^2 - y = 0 \quad y^2 - x = 0 \quad y(y^2 - 1) = 0 \quad y = 0, 1$$

$$(0, 0) \quad (1, 1)$$

$$x = 0, 1$$

$$F(0, 0) = -3hk, \quad F(1, 1) = 3(h^2 - hk + k^2) > 0$$

$$\therefore (1, 1) \text{ において極小値 } -1$$

P. 58

$$4.4 \quad f(x, y) = \begin{vmatrix} \sin x & 0 & \sin y \\ 1 & 1 & -\cos y \\ -\cos x & 1 & 1 \end{vmatrix} = \sin x + \sin y + \sin y \cos x + \cos y \sin x$$

$$f_x(x, y) = \begin{vmatrix} \cos x & 0 & \sin y \\ 0 & 1 & -\cos y \\ \sin x & 1 & 1 \end{vmatrix} = \cos x - \sin x \sin y + \cos x \cos y$$

$$f_y(x, y) = \begin{vmatrix} \sin x & 0 & \cos y \\ 1 & 1 & \sin y \\ -\cos x & 1 & 0 \end{vmatrix} = \cos y + \cos x \cos y - \sin x \sin y$$

$$\cos x + \cos(x+y) = 0$$

$$\cos x - \cos y = 0$$

$$\cos y + \cos(x+y) = 0$$

$$\sin \frac{x+y}{2} \sin \frac{y-x}{2} = 0$$

$$\therefore x+y = 2n\pi \quad y-x = 2k\pi$$

$$y = 2k\pi - x \quad \text{or } k \geq \cos x + \cos(2n\pi) = 0 \quad x = (2m-1)\pi \quad y = (2k-1)\pi$$

$$y = 2n\pi + x \quad \text{or } k \leq \cos x + \cos 2x = 0 \quad 2\cos^2 x + \cos x - 1 = 0$$

$$(2\cos x - 1)(\cos x + 1) = 0 \quad x = \pm \frac{\pi}{3}, \quad x = (2k-1)\pi$$

$$\therefore ((2m-1)\pi, (2k-1)\pi), ((2n \pm \frac{1}{3})\pi, (2m \pm \frac{1}{3})\pi)$$

$$f_{xx} = -\sin x - \sin(x+y)$$

$$f_{xy} = -\sin(x+y)$$

$$f_{yy} = -\sin y - \sin(x+y)$$

$$D = f_{xx} f_{yy} - f_{xy}^2$$

$$= \{\sin x + \sin(x+y)\} \{\sin y + \sin(x+y)\} - \sin^2(x+y)$$

$$= \sin x \sin y + (\sin x + \sin y) \sin(x+y)$$

$$\therefore ((2m-1)\pi, (2k-1)\pi) \text{ or } k \geq \text{極小值 } 0$$

$$((2n + \frac{1}{3})\pi, (2k + \frac{1}{3})\pi) \text{ or } k \leq \text{極大值 } \frac{3\sqrt{3}}{2}$$

$$((2n - \frac{1}{3})\pi, (2k - \frac{1}{3})\pi) \text{ or } k \leq \text{極小值 } -\frac{3\sqrt{3}}{2}$$

$$4.5 \quad x^2 + xy + y^2 = 3$$

$$(1) \quad 2x + y + (x+2y)y' = 0 \quad y' = \frac{-(2x+y)}{x+2y}$$

$$2 + y' + (1+2y')y' + (x+2y)y'' = 0, \quad 2 - 2\frac{2x+y}{x+2y} + 2\left(\frac{2x+y}{x+2y}\right)^2 + (x+2y)y'' = 0$$

$$(x+2y)y'' = \frac{-2}{(x+2y)^2} \{(x+2y)^2 - (2x+y)(x+2y) + (2x+y)^2\}$$

$$= \frac{-2}{(x+2y)^2} (3x^2 + 3xy + 3y^2) = \frac{-6}{(x+2y)^2}$$

$$y'' = \frac{-6}{(x+2y)^3}$$

$$(2) \quad 2x + y = 0 \quad y = -2x \quad x^2 + 2x^2 + 4x^2 = 3 \quad x = \pm 1 \quad y = \mp 2$$

$$x = -1 \text{ or } k \leq \text{極大值 } y = 2$$

$$x = 1 \text{ or } k \leq \text{極小值 } y = -2$$

P.58

4.6 $f(x, y) = x^3 - 3axy + y^3$

(1) $f_x = 3x^2 - 3ay$

$x^2 - ay = 0$

$y^2 - ax = 0$

$f_y = 3y^2 - 3ax$

$(x-y)(x+y+a) = 0$

$x=y=0$

$y = -x-a$

$(0, 0), (a, a)$

$x=y=a$

$x^2 + ax + a^2 = 0$

$f_{xx} = 6x \quad f_{yy} = 6y \quad f_{xy} = -3a$

$D = 36xy - 9a^2$ $x < 0, y < 0 \quad (0, 0)$ $D < 0$ 極大値 $f(0, 0) = 0$

$a < 0$ $x < 0, y < 0 \quad (a, a)$ $f_{xx} = 6a < 0$ 極大値 $-a^3$

$a > 0$ $x > 0, y > 0 \quad (a, a)$ $f_{xx} = 6a > 0$ 極小値 $-a^3$

(2) $3x^2 - 3ay - (3ax - 3y^2)y' = 0$

$y' = \frac{x^2 - ay}{ax - y^2}$

$x^2 - ay = 0$

$y = \frac{x^2}{a}$

$x^3 - 3ax \frac{x^2}{a} + \frac{x^6}{a^3} = 0$

$-2x^3 + \frac{x^6}{a^3} = 0$

$x = 0 \quad \sqrt[3]{2} a$

$6x - 3ay' - (3a - 6y^2)y'' - (3ax - y^2)y''' = 0$ $y = 0 \quad y = \sqrt[3]{2} a$

$y'' = \frac{2x}{ax - y^2}$

$\frac{2\sqrt[3]{2} a}{\sqrt[3]{2} a^2 - 2\sqrt[3]{2} a^2} = -\frac{2}{a}$

$0 < a$

$x = \sqrt[3]{2} a$ 極大値 $\sqrt[3]{4} a$

$a < 0$

$x = \sqrt[3]{2} a$ 極小値 $\sqrt[3]{4} a$

4.7 (1) $f(x, y) = \frac{1}{2} \{ (x-1)^2 + y^2 \} + \sin \sqrt{x^2 + y^2}$

$f_x = x - 1 + \frac{x}{\sqrt{x^2 + y^2}} \cos \sqrt{x^2 + y^2}$

$f_y = y + \frac{y}{\sqrt{x^2 + y^2}} \cos \sqrt{x^2 + y^2}$

$y (1 + \frac{1}{\sqrt{x^2 + y^2}} \cos \sqrt{x^2 + y^2}) = 0 \quad y = 0$ or $\frac{1}{\sqrt{x^2 + y^2}} \cos \sqrt{x^2 + y^2} = -1$

$f_x = 0 \Rightarrow x = 1 \pm \cos \theta \quad x - 1 - x = 0$ 不成立

$\therefore y = 0 \quad x - 1 + \frac{x}{|x|} \cos |x| = 0$

$g(x) = x - 1 \pm \cos x \quad x < 0 \quad g'(x) = 1 \mp \sin x \geq 0 \quad \therefore g(x)$ は増加

$\therefore g(x) = 0$ は $x = 0$ の解 $x = 0$ 以外 $x \rightarrow +\infty$ $x - 1 + \frac{x}{|x|} \cos |x| \rightarrow 0$

$\therefore f_x = 0, f_y = 0$ の解は $(0, 0)$ のみである。

(2) $\cos x = \cos x \quad (\sin x)' = -\sin x$

$\sin x = x - \frac{x^3}{6} \sin \theta(x) \quad 0 < \theta < 1$

P.58

$$\frac{x^2}{2} \sin \theta x = x - \sin x \quad \therefore |x - \sin x| = \frac{x^2}{2} |\sin \theta x| \leq \frac{x^2}{2}$$

$$\therefore -\frac{x^2}{2} \leq x - \sin x \leq \frac{x^2}{2} \quad \therefore x \leq \sin x + \frac{x^2}{2}$$

(3) (2) 5)

$$\sqrt{x^2+y^2} \leq \sin \sqrt{x^2+y^2} + \frac{1}{2}(x^2+y^2)$$

$$\sqrt{x^2+y^2} - x + \frac{1}{2} \leq \frac{1}{2} \{ (x-1)^2 + y^2 \} + \sin \sqrt{x^2+y^2}$$

$$\sqrt{x^2+y^2} - x \geq 0$$

$$\therefore \frac{1}{2} \leq \frac{1}{2} \{ (x-1)^2 + y^2 \} + \sin \sqrt{x^2+y^2}$$

等号が成り立つのは $(0,0)$ のときのみ 極小値 $f(0,0) = \frac{1}{2}$

$$2.8 \quad x^2 + y^2 - 2xy - 2y - 1 = 0 \quad 2x - 2y + (2y - 2x - 2)y' = 0$$

$$y' = \frac{-(2-y)}{y-x-1} \quad y' = 0 \quad y = x = -\frac{1}{2}$$

$$1 - y' + (y'-1)y' + (y-x-1)y'' = 0 \quad y'' = \frac{-1}{y-x-1} > 0$$

$$\therefore x = -\frac{1}{2} \text{ のとき 極小値 } y = -\frac{1}{2}$$

$$2.9 \quad f(x,y) = x^2 + ay + by^2 + cy^3$$

$$f_x(x,y) = 2x \quad x = 0$$

$$f_y(x,y) = a + 2by + 3cy^2 \quad 3cy^2 + 2by + a = 0$$

$$b^2 - 3ac \leq 0$$

$$f_{xx}(x,y) = 2$$

$$f_{xy}(x,y) = 0$$

$$f_{yy}(x,y) = 2b + 6cy$$

$$D = 2(b + 3cy) > 0 \quad b + 3cy > 0$$

$$b^2 \leq 3ac; \quad b < 0, \quad c = 0; \quad a \neq 0, \quad b = c = 0$$

$$2.10 \quad f(x,y) = (x+y)^2 + \frac{3(x+y)}{xy} = (x+y)^2 + 3\left(\frac{1}{x} + \frac{1}{y}\right)$$

$$f_x = 2(x+y) - \frac{3}{x^2}$$

$$y = \pm x \quad y = x \quad 4x^3 - 3 = 0 \quad x = y = \sqrt[3]{\frac{3}{4}}$$

$$f_y = 2(x+y) - \frac{3}{y^2}$$

$$y = -x \quad \checkmark$$

$$f_{xx} = 2 + \frac{6}{x^3}$$

$$f_{xy} = 2$$

$$f_{yy} = 2 + \frac{6}{y^3}$$

$$D = f_{xx} f_{yy} - f_{xy}^2 > 0 \quad f_{xx} > 0$$

$$\left(\sqrt[3]{\frac{3}{4}}\right)^2, \left(\sqrt[3]{\frac{3}{4}}\right)^2 \text{ のとき 極小値 } 9\sqrt[3]{\frac{3}{4}}$$

P.59

§.5. 偏微分の応用2 (最大・最小)

5.1 (1) $x+y+z=9$ $f(x,y,z)=xyz$

$$z=9-x-y \quad \therefore f=xy(9-x-y)=9xy-x^2y-xy^2$$

$$f_x=9y-2x^2y-y^2 \quad y(9-2x-y)=0 \quad x=y=0$$

$$f_y=9x-x^2-2xy \quad x(9-x-2y)=0 \quad x=0 \quad y=9 \quad y=0 \quad x=9$$

$$(0,0), (9,0), (0,9), (3,3)$$

$$-9+3y=0 \quad y=x=3$$

$$f_{xx}=-2y \quad f_{xy}=9-2x-2y \quad f_{yy}=-2x$$

$$D=f_{xx}f_{yy}-f_{xy}^2=4xy-(9-2x-2y)^2$$

$$(0,0) \text{ のとき } D < 0. \quad (9,0), (0,9) \text{ のとき } D < 0$$

$$(3,3) \text{ のとき } D=36-9 > 0 \quad f_{xx}=-6 < 0$$

$$(3,3) \text{ で } \text{極大値 } 27 \text{ (最大値)}$$

(2) $x^2+y^2=1$; $f=x^2+2xy+y^2$

$$2x+2y-2\lambda x=0$$

$$y=(\lambda-1)x$$

$$y(1-\lambda)^2=0$$

$$2y+2x-2\lambda y=0$$

$$x=\lambda-1y$$

$$y=0, \lambda=0, \lambda=2$$

$$x=0 \quad y=-x$$

$$x=y$$

$$\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right) \text{ のとき } \text{最小値 } 0$$

$$x=\pm\frac{1}{\sqrt{2}}, y=\mp\frac{1}{\sqrt{2}} \quad x-y=\pm\frac{2}{\sqrt{2}}$$

$$\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) \text{ のとき } \text{最大値 } 2$$

5.2

$$x^2+y^2+z^2 \leq 1 \quad f=x^2+y^2-2y+z+2x$$

$$x^2+y^2+z^2=1 \text{ とおくと } z_x=-\frac{x}{z} \quad z_y=-\frac{y}{z}$$

$$f_x=2x-y+z+(y+x)\left(-\frac{x}{z}\right)=0 \quad 2x^2y-y^2+z^2y-(2yx-x^2+2x)=0$$

$$f_y=2y-x+z+(x+y)\left(-\frac{y}{z}\right)=0 \quad x^2y^2+z(y-x)=0$$

$$y=x \quad x+z-\frac{2x^2}{z}=0 \quad z^2+xz-2x^2=0 \quad (z+2x)(z-x)=0$$

$$x=0 \quad z=-2x \quad x=y=z=\frac{1}{\sqrt{3}} \quad x=y=\frac{1}{\sqrt{2}} \quad z=\frac{1}{\sqrt{2}}$$

$$x+y=z \quad 2x-y+z-x=0 \quad z=y-x$$

$$2y-x+z-y=0 \quad z=x-y$$

$$\therefore x=y=0$$

$$(0,0) \quad \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \quad \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$f=\frac{1}{3}+\frac{1}{3}-\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1 \text{ 最大値}$$

$$f=\frac{1}{2}+\frac{1}{2}-\frac{1}{2}-\frac{2}{2}-\frac{2}{2}=-\frac{1}{2} \text{ 最小値}$$

P 59

5.3 $x, y, z > 0 \quad x+y+z=1 \quad \text{例 2}$

$$H(x, y, z) = -(x \log x + y \log y + z \log z)$$

$$(z = 1-x-y) \quad H = -x \log x - y \log y - (1-x-y) \log(1-x-y)$$

$$H_x = -\log x - \frac{1}{\log 2} + \log(1-x-y) - (1-x-y) \frac{-1}{1-x-y} \frac{1}{\log 2}$$

$$= \log \frac{1-x-y}{x} \quad \begin{cases} 1-x-y=x \\ 1-x-y=y \end{cases} \quad x=y=z=\frac{1}{3}$$

$$H_y = \log \frac{1-x-y}{y}$$

$$H_{xx} = -\frac{1}{x \log 2} - \frac{1}{(1-x-y) \log 2} \quad H_{xy} = \frac{-1}{(1-x-y) \log 2}$$

$$H_{yy} = -\frac{1}{y \log 2} - \frac{1}{(1-x-y) \log 2}$$

$$D = H_{xx} H_{yy} - H_{xy}^2 = \frac{1}{(\log 2)^2} \left(\frac{1}{xy} + \left(\frac{1}{x} + \frac{1}{y} \right) \frac{1}{1-x-y} \right)$$

$$\left(\frac{1}{3}, \frac{1}{3} \right) \quad z = \frac{1}{3} \quad D > 0 \quad H_{xx} < 0 \quad \left(\frac{1}{3}, \frac{1}{3} \right) \quad z = \frac{1}{3} \quad \text{极大值} \quad \log_2 3$$

5.4 $\frac{x^2}{4} + y^2 \leq 1 \quad z = x + \frac{y}{2} + \sqrt{1 - \frac{x^2}{4} - y^2}$

$$\frac{x^2}{4} + y^2 = r^2 \quad \text{极小} \quad |V| \leq 1 \quad r \frac{\partial r}{\partial x} = \frac{x}{4}$$

$$z_x = 1 - \frac{r}{\sqrt{1-r^2}} \frac{\partial r}{\partial x} = 1 - \frac{1}{\sqrt{1-r^2}} \frac{x}{4} \quad r \frac{\partial r}{\partial y} = \frac{y}{2}$$

$$4\sqrt{1-r^2} = x \quad x = 4y$$

$$z_y = 1 - \frac{r}{\sqrt{1-r^2}} \frac{\partial r}{\partial y} = 1 - \frac{1}{\sqrt{1-r^2}} y \quad \sqrt{1-r^2} = y$$

$$4y^2 + y^2 = r^2 \quad y = \pm \frac{1}{\sqrt{5}} r \quad x = \pm \frac{4}{\sqrt{5}} r$$

$$\left(\frac{4}{\sqrt{5}} r, \frac{1}{\sqrt{5}} r \right) \text{ 极大值} \quad z = \sqrt{5} r + \sqrt{1-r^2} \quad \frac{\partial z}{\partial r} = 0 \quad \sqrt{5}$$

$$\left(-\frac{4}{\sqrt{5}} r, \frac{1}{\sqrt{5}} r \right) \text{ 极大值} \quad z = -\sqrt{5} r + \sqrt{1-r^2} \quad \frac{\partial z}{\partial r} = 0 \quad -\sqrt{5}$$

5.5 $m, n, p > 0 \quad x+y+z=a \quad (\text{5.4 例 5.1})$

$$V = x^m y^n z^p = x^m y^n (a-x-y)^p$$

$$V_x = \{ m(a-x-y) - px \} x^{m-1} y^n (a-x-y)^{p-1}$$

$$V_y = \{ n(a-x-y) - py \} x^m y^{n-1} (a-x-y)^{p-1}$$

$$ma - (m+p)x - m y = 0$$

$$na - nx - (m+py) = 0$$

$$x = \frac{m a}{m+n+p}$$

$$y = \frac{n a}{m+n+p}$$

$$z = a - x - y = \frac{p a}{m+n+p}$$

$$\therefore V = m^m n^n p^p \frac{1}{(m+n+p)^{m+n+p}}$$

P.59

5.6 (5.5 と 5.61)

(x, y) が $x^2 + y^2 = 1$ のとき $f(x, y) = ax^2 + 2bxy + cy^2$ の最大値、最小値

$$\begin{cases} ax + by - \lambda x = 0 \\ bx + cy - \lambda y = 0 \end{cases} \quad \begin{cases} (a-\lambda)x + by = 0 \\ bx + (c-\lambda)y = 0 \end{cases} \quad \text{かつ } \omega \text{ の 上 に 外 に 解 を 求 め る 時}$$

$$\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = 0 \quad \therefore \text{この解を } \lambda \text{ とすると}$$

$$ax^2 + 2bxy + cy^2 - \lambda(x^2 + y^2) = 0 \quad \therefore ax^2 + 2bxy + cy^2 = \lambda$$

2次曲線 $x^2 + y^2 = 1$ と $ax^2 + 2bxy + cy^2 = \lambda$ が共通点をもつ接線を用いて最大値、最小値を求めよ

5.7 (5.6 と 5.71)

$$0 \leq x \leq 1 \quad y = e^x \quad y = ax + b$$

$$\begin{aligned} I(a, b) &= \int_0^1 \{e^{2x} - (ax+b)^2\} dx = \int_0^1 \{e^{2x} - 2(ax+b)e^x + (ax+b)^2\} dx \\ &= \left[\frac{1}{2} e^{2x} - 2(ax+b)e^x + 2ae^x + \frac{1}{3a} (ax+b)^3 \right]_0^1 \\ &= \frac{1}{2}(e^2-1) - 2(a+b)e + 2b + 2ae - 2a + \frac{1}{3a} \{(a+b)^3 - b^3\} \\ &= \frac{1}{2}(e^2-1) - 2be + 2b - 2a + \frac{1}{3}a^2 + ab + b^2 \end{aligned}$$

$$I_a = -2 + \frac{2}{3}a + b = 0 \quad -b + 2a + 3b = 0 \quad -b - 4 + 4e - a = 0$$

$$I_b = 2 - 2e + a + 2b = 0 \quad 2 - 2e + a + 2b = 0 \quad \begin{cases} b = 4e - 10 \\ a = -6e + 18 \end{cases}$$

5.8

$$E = \sum_{i=1}^5 \{y_i - (ax_i + b)\}^2 = \sum_{i=1}^5 y_i^2 - 2 \sum_{i=1}^5 y_i(ax_i + b) + \sum_{i=1}^5 (ax_i + b)^2$$

$$E_a = -2 \sum y_i x_i + 2 \sum (ax_i + b)x_i$$

$$E_b = -2 \sum y_i + 2 \sum (ax_i + b)$$

$$\therefore a \sum x_i^2 + b \sum x_i = \sum x_i y_i$$

$$a \sum x_i + 5b = \sum y_i$$

$$a \sum x_i^2 + 5b\bar{x} = \sum x_i y_i$$

$$a\bar{x} + b = \bar{y}$$

$$a = \frac{\sum x_i y_i - 5\bar{x}\bar{y}}{\sum x_i^2 - 5\bar{x}^2}$$

$$b = \frac{\bar{y} \sum x_i^2 - \bar{x} \sum x_i y_i}{\sum x_i^2 - 5\bar{x}^2}$$

p. 59

5.9 (5.8 E 5.9 I)

$$V = a^2 h = 4000$$

$$S = a^2 + 4ah$$

$$\therefore V = a^2 + \frac{16,000}{a}$$

$$V_a = 2a - \frac{16,000}{a^2} \quad a^3 - 8000 = 0 \quad a = 20$$

$$\therefore a = 20 \quad h = 10$$

5.10 (5.9 E 5.10 I)

$$x + y + z = a \quad z = a - x - y$$

$$V = xyz = axy - x^2y - xy^2$$

$$V_x = ay - 2xy - y^2 \quad y(a - 2x - y) = 0$$

$$V_y = ax - x^2 - 2xy \quad x(a - x - 2y) = 0$$

$$(0, 0) \quad (0, a) \quad (a, 0) \quad \left(\frac{a}{3}, \frac{a}{3}\right)$$

$$V_{xx} = -2y \quad V_{yy} = -2x \quad V_{xy} = a - 2x - 2y$$

$$D = V_{xx} V_{yy} - V_{xy}^2 = 4xy - (a - 2x - 2y)^2$$

$$(0, 0), (0, a), (a, 0) \text{ のとき } D < 0$$

$$\left(\frac{a}{3}, \frac{a}{3}\right) \text{ のとき } D > 0 \quad V_{xx} < 0$$

$$\therefore \left(\frac{a}{3}, \frac{a}{3}\right) \text{ のとき } \frac{a^3}{27} \text{ 最大値}$$

P.59

§ 6. 偏微分の応用 (図形)

6.1 (1) $f(x,y) = x^2 + y^2 - 3xy$

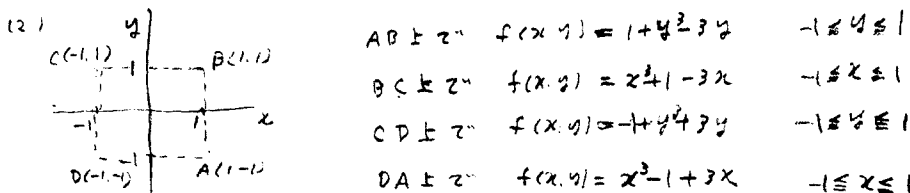
$f_x = 2x - 3y = 0 \quad x^2 - y = 0 \quad (0,0) \quad (1,1)$

$f_y = 2y - 3x = 0 \quad y^2 - x = 0$

$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = -3$

$D = 36xy - 9 \quad (0,0) \text{ の } D < 0 \quad (1,1) \text{ の } D > 0$

$f_{xx} > 0 \quad \therefore (1,1) \text{ で 極小値 } -1$



AB上 $z = \frac{dz}{dy} = 2y - 3 < 0$ 減少 $\text{Max: } f(A) = 3 \quad \text{min: } f(B) = -1$

BC上 $z = \frac{dz}{dx} = 2x - 3 < 0$ 減少 $\text{Max: } f(C) = 3 \quad \text{min: } f(D) = -1$

CD上 $z = \frac{dz}{dy} = 2y + 3 > 0$ 増加 $\text{Max: } f(C) = 3 \quad \text{min: } f(D) = -5$

DA上 $z = \frac{dz}{dx} = 2x + 3 > 0$ 増加 $\text{Max: } f(A) = 3 \quad \text{min: } f(D) = -5$

(3) $|x| < 1, |y| < 1$ において $f(x,y)$ が最大値または最小値をとるとき

それは極大値または極小値と相違ないから、(1)より極値は $f(1,1)$

のみである。故に 最大値は $f(1,1) = f(-1,1) = 3$

最小値は $f(-1,-1) = -5$

6.2 $P_i(x_i, y_i) \quad P(x, y)$

$V = \sum_{i=1}^n [(x_i - x)^2 + (y_i - y)^2]$

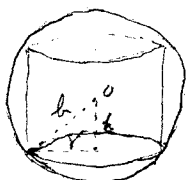
$V_x = -2 \sum_{i=1}^n (x_i - x) \quad V_y = -2 \sum_{i=1}^n (y_i - y) \quad x = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$

$y = \frac{1}{n} \sum_{i=1}^n y_i = \bar{y}$

$V_{xx} = 2n \quad V_{yy} = 2n \quad V_{xy} = 0$

$D = V_{xx} V_{yy} - V_{xy}^2 = 4n^2 > 0 \quad \therefore \text{最小値 } V(\bar{x}, \bar{y})$

6.3



$r^2 + h^2 = t^2 \quad 0 \leq t \leq R$

$V = 2\pi r^2 h, \quad S = 2\pi r^2 + 2\pi r h$

$F = \frac{S}{V} = \frac{1}{h} + \frac{1}{r}$

$\frac{2}{(2\lambda)^{3/2}} = t^2 \quad \left(\frac{2}{t^2}\right)^3 = (2\lambda)^2$

$\frac{2}{t^6} = \lambda^2 \quad \lambda = \frac{\sqrt{2}}{t^3}$

$-\frac{1}{\lambda^2} + 2\lambda h = 0 \quad -\frac{1}{t^3} + 2\lambda r = 0$

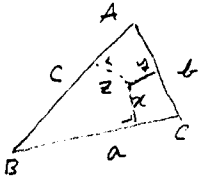
$h = r = \sqrt[3]{\frac{1}{2\lambda}} = \frac{t}{\sqrt{2}}$

$\therefore F = \frac{2\sqrt{2}}{t}$

P 59. 60

1. $k = a$ のとき 最小 $(1, 1, 1)$ 底面の半径: 高さ = $1:2$

6.4



$$ax + by + cz = k \quad (\text{一定}) \quad 2S = k$$

$$V = xyz = xy \frac{1}{c} (k - ax - by) \\ = \frac{1}{c} (kxy - ax^2y - by^2x)$$

$$\sqrt{x} = \frac{1}{c} (ky - 2axy - by^2) = \frac{y}{c} (k - 2ax - by) = 0$$

$$\sqrt{y} = \frac{1}{c} (kx - ax^2 - 2by) = \frac{x}{c} (k - ax - 2by) = 0$$

$$(0, 0), \left(\frac{k}{a}, 0\right), \left(0, \frac{k}{b}\right), \left(\frac{k}{3a}, \frac{k}{3b}\right)$$

$$\sqrt{x}x = -\frac{2ay}{c} \quad \sqrt{y}y = -\frac{2bx}{c} \quad \sqrt{xy} = \frac{1}{c} (k - 2ax - 2by)$$

$$D = \sqrt{x}x \sqrt{y}y - \sqrt{xy}^2 = \frac{1}{c^2} \{4abxy - (k - 2ax - 2by)^2\}$$

$$(0, 0), \left(\frac{k}{a}, 0\right), \left(0, \frac{k}{b}\right) \text{ とき } D < 0$$

$$\left(\frac{k}{3a}, \frac{k}{3b}\right) \text{ とき } D > 0 \quad \sqrt{x}x < 0.$$

$$\therefore V(x, y) = \frac{1}{c} \frac{k^2}{3a \cdot 3b} \cdot \frac{k}{3} = \frac{k^3}{27abc} = \frac{8S^3}{27abc}$$

6.5 (1) $\begin{cases} x+y+z=6 & 2(xy+yz+zx)=18 \\ x^2+y^2+z^2=9 & xy+(x+y)(6-(x+y))=9 \\ & (x+y)^2-(x+y)+9=0 \end{cases}$

$$y^2 + (x-6)y + x^2 - 6x + 9 = 0 \quad y^2 + (x-6)y + (x-3)^2 = 0$$

$$\therefore (x-6)^2 - 4(x-3)^2 \geq 0 \quad (x-6-2x+6)(x-6+2x-6) \geq 0$$

$$-x(3x-12) \geq 0 \quad x(x-4) \leq 0 \quad 0 < x \leq 4$$

(2) $x+y+z=6$ より $y+z=6-x$

$$x^2+y^2+z^2=9$$
 より $x(6-x)+y^2=9 \quad \therefore yz=9-6x-x^2=(x-3)^2$

$$\therefore V = xyz$$

$$= x(x-3)^2$$

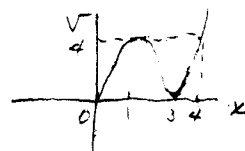
$$V' = (x-3)^2 + 2x(x-3) = (x-3)(3x-3) = 3(x-3)(x-1)$$

$\therefore V$ は $0 < x \leq 1$ で増加し $1 < x \leq 3$ で減少し $3 < x < 4$ で増加

$$\therefore V(1) = 4 \quad V(3) = 4$$

\therefore 最大値 $(1, 1, 4)$ $(1, 4, 1)$ 或 $(4, 1, 1)$

で 4



P.60

6.6 $xy + yz + zx = \frac{1}{2}$

$$V = xyz = xy \frac{1}{x+y} (\frac{1}{2} - xy) = \frac{1}{x+y} (\frac{1}{2}xy - x^2y^2)$$

$$V_x = \frac{1}{x+y} (\frac{1}{2}y - 2xy^2) - \frac{1}{(x+y)^2} (\frac{1}{2}xy - x^2y^2) = \frac{1}{(2+y)^2} (-x^2y^2 + \frac{1}{2}y^2 - 2xy^3)$$

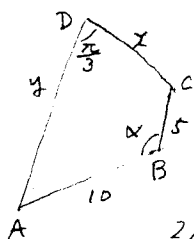
$$V_y = \frac{1}{(x+y)^2} (-x^2y^2 + \frac{1}{2}x^2 - 2x^2y)$$

$$y^2(-x^2 + \frac{1}{2} - 2xy) = 0 \quad (0,0) \quad (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$

$$x^2(-y^2 + \frac{1}{2} - 2xy) = 0$$

$\therefore V(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{1\sqrt{2}}$ 最大值

6.7



$$AC^2 = 100 + 25 - 100 \cos \alpha$$

$$AC^2 = x^2 + y^2 - 2xy \cos \frac{\pi}{3}$$

$$125 - 100 \cos \alpha = x^2 + y^2 - xy$$

$$\therefore 100 \sin \alpha \frac{\partial \alpha}{\partial x} = 2x - y \quad 100 \sin \alpha \frac{\partial \alpha}{\partial y} = 2y - x$$

$$25 = xy \sin \frac{\pi}{3} + 50 \sin \alpha$$

$$= \frac{\sqrt{3}}{2} xy + 50 \sin \alpha$$

$$\frac{\partial(25)}{\partial x} = \frac{\sqrt{3}}{2} y + 50 \cos \alpha \frac{\partial \alpha}{\partial x} = \frac{\sqrt{3}}{2} y + \frac{50 \cos \alpha}{100 \sin \alpha} (2x - y)$$

$$= \frac{\sqrt{3}}{2} y + \frac{1}{2} \cot \alpha (2x - y)$$

$$\frac{\partial(25)}{\partial y} = \frac{\sqrt{3}}{2} x + \frac{1}{2} \cot \alpha (2y - x)$$

$$\begin{cases} x \cot \alpha + y (\frac{\sqrt{3}}{2} - \frac{1}{2} \cot \alpha) = 0 \\ y \cot \alpha + x (\frac{\sqrt{3}}{2} - \frac{1}{2} \cot \alpha) = 0 \end{cases} = \text{除非 } (0,0) \text{ 以外无解存在}$$

$$\cot^2 \alpha - (\frac{\sqrt{3}}{2} - \frac{1}{2} \cot \alpha)^2 = 0 \quad \cot^2 \alpha + \frac{2}{\sqrt{3}} \cot \alpha - 1 = 0$$

$$\cot \alpha = -\frac{1}{\sqrt{3}} \pm \frac{2}{\sqrt{3}} \quad \frac{1}{\sqrt{3}}, -\sqrt{3} \quad \alpha = \frac{\pi}{3}, \frac{5}{6}\pi$$

i) $\alpha = \frac{\pi}{3}$ 时 $25 = x^2 + y^2 - xy \quad 25 = \frac{\sqrt{3}}{2} xy + 25\sqrt{3}$

$$\frac{\sqrt{3}}{2} y + \lambda(2x - y) = 0$$

$$2\lambda x + (\frac{\sqrt{3}}{2} - \lambda)y = 0$$

$$\frac{\sqrt{3}}{2} x + \lambda(2y - x) = 0$$

$$(\frac{\sqrt{3}}{2} - \lambda)x + 2\lambda y = 0$$

除非 $(0,0)$ 以外的解存在

$$(\frac{\sqrt{3}}{2} - \lambda)^2 - 4\lambda^2 = 0 \quad (\frac{\sqrt{3}}{2} - 3\lambda)(\frac{\sqrt{3}}{2} + \lambda) = 0 \quad \lambda = \frac{1}{2\sqrt{3}} \text{ 或 } -\frac{\sqrt{3}}{2}$$

$$\lambda = \frac{1}{2\sqrt{3}} \text{ 时 } \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} = 0 \quad y = -x \quad x, y > 0 \text{ 不成立}$$

$$\lambda = \frac{-\sqrt{3}}{2} \text{ 时 } -x + y = 0 \quad y = x \quad x^2 = 25 \quad x = y = 5\sqrt{3}$$

