P.61 第7章 重積分

§1. 重積分の計算

1.1 (1) $\int_0^\pi \int_0^{\frac{\pi}{2}} x\sin(x+y)\,dx\,dy = \int_0^\pi \left[-x\cos(x+y)+\sin(x+y)\right]_0^{\frac{\pi}{2}} dy$

$= \int_0^\pi \left(-\frac{\pi}{2}\cos\left(\frac{\pi}{2}+y\right)+\sin\left(\frac{\pi}{2}+y\right)-\sin y\right)dy$

$= \left[-\frac{\pi}{2}\sin\left(\frac{\pi}{2}+y\right) - \cos\left(\frac{\pi}{2}+y\right)+\cos y\right]_0^\pi$

$= \frac{\pi}{2}-1-\left(-\frac{\pi}{2}+1\right) = \pi-2$

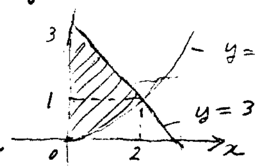
(2) $\int_0^1 dx\int_0^1 \sin^2\pi xy\,dy = \int_0^1 dx \int_0^1 \frac{1-\cos 2\pi xy}{2}dy = \int_0^1 \left[\frac{y}{2} - \frac{\sin 2\pi xy}{4\pi}\right]_0^1 dx$

$=\int_0^1 \left(\frac{1}{2}\right)dx = \frac{1}{2}$

(3) $\int_0^2 dx\int_0^x xy^2\,dy = \int_0^2 \left[x\frac{y^3}{3}\right]_0^x dx = \int_0^2 \frac{x^4}{3}dx = \frac{1}{3}\left[\frac{x^5}{5}\right]_0^2 = \frac{32}{15}$

1.2 (1) $\int_0^2 dx\int_{\frac{x^2}{4}}^{3-x} f(x,y)$

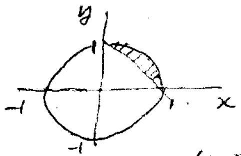
$= \int_0^1 dy \int_0^{2\sqrt{y}} f(x,y)\,dx + \int_1^3 dy\int_0^{3-y} f(x,y)\,dx$



$y=\frac{x^2}{4}$ $\quad x=2\sqrt{y}$

$y=3-x$ $\quad x=3-y$

1.3 (1) $\iint_D (x^2+y^2)\,dx\,dy$ $\quad D:\ x^2+y^2\leq 1 \quad x+y\geq 1$



$=\int_0^1 dx\int_{1-x}^{\sqrt{1-x^2}}(x^2+y^2)\,dy$

$=\int_0^1 \left[x^2y+\frac{y^3}{3}\right]_{1-x}^{\sqrt{1-x^2}}dx = \int_0^1 \left[\left\{\left(x^2+\frac{1}{3}(1-x^2)\right)\sqrt{1-x^2}\right\} - \left\{x^2(1-x)+\frac{(1-x)^3}{3}\right\}\right]dx$

$=\int_0^1 \left\{\frac{1}{3}(1+2x^2)\sqrt{1-x^2} - \left(\frac{1}{3} - x+2x^2-\frac{4x^3}{3}\right)\right\}dx$

$=\frac{1}{3}\int_0^1 (1+2x^2)\sqrt{1-x^2}\,dx - \left[\frac{1}{3}x-\frac{1}{2}x^2+\frac{2}{3}x^3-\frac{x^4}{3}\right]_0^1$

$x=\sin t$ とおく

$=\frac{1}{3}\int_0^{\frac{\pi}{2}}(1+2\sin^2t)\cos^2t\,dt - \frac{1}{6}$

$=\frac{1}{3}\int_0^{\frac{\pi}{2}}(1+\sin^2t-2\sin^4t)\,dt-\frac{1}{6}$

$=\frac{1}{3}\left(\frac{\pi}{2}+\frac{\pi}{4}-\frac{3}{2}\cdot\frac{\pi}{4}\right)-\frac{1}{6} = \frac{\pi}{8}-\frac{1}{6}$

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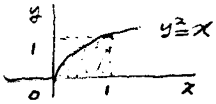
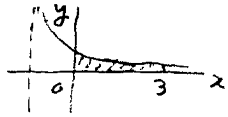
1.3 (2) $\iint_D xy\,dx\,dy$ D : $y \leqq \sqrt{x}$, $y \geqq 0$ $x \leqq 1$



$= \int_0^1 dx \int_0^{\sqrt{x}} xy\,dy = \int_0^1 \left[\frac{x}{2}y^2\right]_0^{\sqrt{x}} dx = \int_0^1 \frac{x^2}{2}dx = \left[\frac{x^3}{6}\right]_0^1 = \frac{1}{6}$

(3) $\iint_D (x+y^2)\,dx\,dy$ D : $0 \leqq x \leqq 3$, $0 \leqq y \leqq \frac{1}{x+1}$



$= \int_0^3 dx \int_0^{\frac{1}{x+1}} (x+y^2)\,dy = \int_0^3 \left[xy + \frac{1}{3}y^3\right]_0^{\frac{1}{x+1}} dx = \int_0^3 \left\{\frac{x}{x+1} + \frac{1}{3}\left(\frac{1}{x+1}\right)^3\right\} dx$

$= \int_0^3 \left(1 - \frac{1}{x+1}\right) dx + \frac{1}{3}\int_0^3 \frac{1}{(x+1)^3}\,dx = \left[x - \log(x+1) - \frac{1}{6(x+1)^2}\right]_0^3$

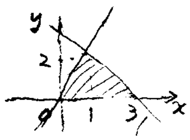
$= 3 - \log 4 - \frac{1}{6\cdot 4^2} + \frac{1}{6} = \frac{101}{32} - \log 4$

(4) $\iint_D (1+x+y)\,dx\,dy$ D : $x \geqq 0$, $y \geqq 0$ $x+y \leqq 1$

$= \int_0^1 dx \int_0^{1-x} (1+x+y)\,dy = \int_0^1 \left[(1+x)y + \frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1 \left\{(1+x)(1-x) + \frac{(1-x)^2}{2}\right\} dx$

$= \left[x - \frac{x^3}{3} - \frac{(1-x)^3}{6}\right]_0^1 = 1 - \frac{1}{3} + \frac{1}{6} = \frac{5}{6}$

(5) $\iint_D x\sqrt{y}\,dx\,dy$ D : $y \leqq 2x$ $y \leqq -x+3$, $y \geqq 0$



$y = 2x$ $y = -x+3$

$= \int_0^2 dy \int_{\frac{y}{2}}^{3-y} x\sqrt{y}\,dx = \int_0^2 \left[\frac{x^2}{2}\sqrt{y}\right]_{\frac{y}{2}}^{3-y} dy$

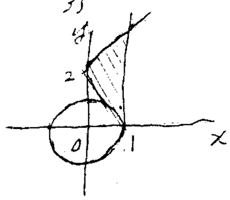
$= \int_0^2 \left(\frac{(3-y)^2}{2}\sqrt{y} - \frac{y^2}{8}\sqrt{y}\right) dy = \int_0^{\sqrt{2}} \left(\frac{(3-t^2)^2}{2}t - \frac{t^4}{8}t\right) 2t\,dt$ $\sqrt{y} = t$, $y = t^2$

$= \int_0^{\sqrt{2}} \left((3-t^2)^2 t^2 - \frac{1}{4}t^6\right) dt = \int_0^{\sqrt{2}} \left(9t^2 - 6t^4 + t^6 - \frac{1}{4}t^6\right) dt$

$= \left[3t^3 - \frac{6}{5}t^5 + \frac{3}{28}t^7\right]_0^{\sqrt{2}} = 6\sqrt{2} - \frac{24}{5}\sqrt{2} + \frac{3\cdot 8}{28}\sqrt{2} = \frac{72}{35}\sqrt{2}$

(6) $\iint_D xy\,dx\,dy$ D : $x^2+y^2 \geqq 1$ $0 \leqq x \leqq 1$

$x - y + 2 \geqq 0$ $y \geqq 0$

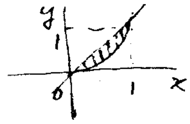


$= \int_0^1 dx \int_{\sqrt{1-x^2}}^{x+2} xy\,dy$

$= \int_0^1 x\left[\frac{y^2}{2}\right]_{\sqrt{1-x^2}}^{x+2} dx = \int_0^1 \left\{\frac{x}{2}(x+2)^2 - \frac{x}{2}(1-x^2)\right\} dx$

$= \int_0^1 \left(x^3 + 2x^2 + \frac{3}{2}x\right) dx = \left[\frac{x^4}{4} + \frac{2}{3}x^3 + \frac{3}{4}x^2\right]_0^1 = \frac{5}{3}$

(7) $\iint \frac{1}{\sqrt{1+x}}\,dx\,dy$ D: $0 \leqq x \leqq 1$, $x^2 \leqq y \leqq x$



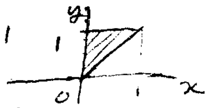
$= \int_0^1 dx \int_{x^2}^{x} \frac{1}{\sqrt{1+x}}\,dy = \int_0^1 \left(\frac{x}{\sqrt{1+x}} - \frac{x^2}{\sqrt{1+x}}\right) dx$ $\quad \sqrt{1+x} = t$

$x = t^2 - 1 \quad dx = 2t\,dt$

$= \int_1^{\sqrt{2}} \left(\frac{t^2-1}{t} - \frac{(t^2-1)^2}{t}\right) 2t\,dt = 2\int_1^{\sqrt{2}} (t^2 - 1 - t^4 + 2t^2 - 1)\,dt$

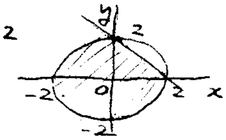
$= 2\left[-\frac{t^5}{5} + t^3 - 2t\right]_1^{\sqrt{2}} = 2\left(-\frac{4\sqrt{2}}{5} + 2\sqrt{2} - 2\sqrt{2} - \left(-\frac{1}{5} + 1 - 2\right)\right) = \frac{12 - 8\sqrt{2}}{5}$

(8) $\iint_D x^2 y\,dx\,dy$ D: $x \geqq 0,\ y \geqq 0 \quad 0 \leqq x \leqq y \leqq 1$



$= \int_0^1 dx \int_x^1 x^2 y\,dy = \int_0^1 \left[\frac{x^2}{2} y^2\right]_x^1 = \int_0^1 \left(\frac{1}{2}x^2 - \frac{x^4}{2}\right) dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{15}$

(9) $\iint_D x\,dx\,dy$ D: $x^2 + y^2 \leqq 4 \quad x + y \leqq 2$



$= \int_{-2}^{0} x\,dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy + \int_0^2 x\,dx \int_{-\sqrt{4-x^2}}^{2-x} dy$

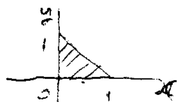
$= \int_{-2}^{0} x\,[y]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\,dx + \int_0^2 x\,[y]_{-\sqrt{4-x^2}}^{2-x}\,dx$

$= \int_{-2}^{0} 2x\sqrt{4-x^2}\,dx + \int_0^2 \left(x(2-x) + x\sqrt{1-x^2}\right) dx$

$= \left[-\frac{2}{3}(4-x^2)^{\frac{3}{2}}\right]_{-2}^{0} + \left[x^2 - \frac{x^3}{3}\right]_0^2 - \left[\frac{1}{3}(4-x^2)^{\frac{3}{2}}\right]_0^2$

$= \frac{-16}{3} + \frac{4}{3} + \frac{8}{3} = -\frac{4}{3}$

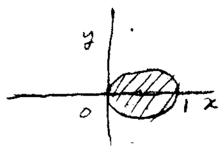
(10) $\iint_D (x^2 + y^2)\,dx\,dy$ D: $x \geqq 0,\ y \geqq 0 \quad x + y \leqq 1$



$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx = \int_0^1 \left(x^2(1-x) + \frac{(1-x)^3}{3}\right) dx$

$= \frac{1}{3} - \frac{1}{4} - \frac{1}{12}\left[(1-x)^4\right]_0^1 = \frac{1}{6}$

1.4 (1) $\iint_D \sqrt{x}\,dx\,dy$ D: $x^2 + y^2 \leqq x \quad \left(x - \frac{1}{2}\right)^2 + y^2 \leqq \frac{1}{4}$



$= \int_0^1 \sqrt{x}\,dx \int_{-\sqrt{\frac{1}{4} - (x-\frac{1}{2})^2}}^{\sqrt{\frac{1}{4} - (x-\frac{1}{2})^2}} dy = \int_0^1 2\sqrt{x}\sqrt{x - x^2}\,dx = 2\int_0^1 x\sqrt{1-x}\,dx$

$\sqrt{1-x} = t$

$1 - t^2 = x$

$= 2\int_1^0 (1-t^2)\,t\,(-2t)\,dt = 4\int_0^1 (t^2 - t^4)\,dt = 4\left(\frac{1}{3} - \frac{1}{5}\right)$

$= \frac{8}{15}$

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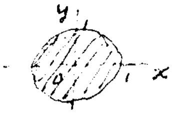
1.4 (2) $\iint_D (1-x^2-y^2)\,dx\,dy$ $\quad$ D: $x^2+y^2 \le 1$ $\quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$



$$= \int_0^{2\pi} d\theta \int_0^1 (1-r^2)\, r\,dr \qquad J = \frac{\partial(x,y)}{\partial(r,\theta)} = r$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = \frac{2\pi}{4} = \frac{\pi}{2}$$

(3) $\iint_D (x^2+y^2)\,dx\,dy$ $\quad$ D: $x^2+y^2 \le a^2$, $0 \le x$, $y \le x$ $\quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^a r^2 r\,dr = \int_0^{\frac{\pi}{2}} \frac{a^4}{4}\,d\theta = \frac{a^4\pi}{8}$$

(4) $\iint_D (1+x)\,dx\,dy$ $\quad$ D: $x^2+y^2 \le 2x$, $y \ge 0$

$$x = r\cos\theta, \quad y = r\sin\theta \quad r \le 2\cos\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} (1+r\cos\theta)\, r\,dr$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2} + \frac{r^3}{3}\cos\theta\right]_0^{2\cos\theta} d\theta = \int_0^{\frac{\pi}{2}} \left(2\cos^2\theta + \frac{8}{3}\cos^4\theta\right) d\theta$$

$$= 2\,\frac{\pi}{4} + \frac{8}{3}\,\frac{3}{4}\cdot\frac{\pi}{4} = \pi \qquad \int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx = \int_0^{\frac{\pi}{2}} \cos^{2n} x\,dx = \frac{2n-1}{2n}\,\frac{2n-3}{2n-2} \cdots \frac{1}{2}\,\frac{\pi}{2}$$

(5) $\iint xy\,dx\,dy$ $\quad$ D: $x^2+y^2 \le 1$ $\quad x \ge 0$, $y \ge 0$ $\quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r^3 \sin\theta\cos\theta\,d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{4}\sin\theta\cos\theta\,d\theta = \frac{1}{8}\left[\sin^2\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{8}$$

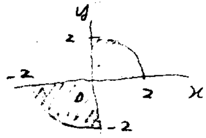
(6) $\iint_D e^{-(x^2+y^2)}\,dx\,dy$ $\quad$ D: $x^2+y^2 \le 1$ $\quad x = r\cos\theta,\ y = r\sin\theta$

$$= \int_0^{2\pi} d\theta \int_0^1 e^{-r^2} r\,dr = \int_0^{2\pi} \left[-\frac{1}{2}e^{-r^2}\right]_0^1 d\theta = \frac{1}{2}(1-e^{-1})\,2\pi = (1-e^{-1})\pi$$

(7) $\iint_D x\,dx\,dy$ $\quad$ D: $x^2+y^2 \le 1$ $\quad x > 0$ $\quad x = r\cos\theta$, $y = r\sin\theta$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^1 r^2\cos\theta\,dr = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3}\cos\theta\,d\theta = \frac{1}{3}\left[\sin\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{3}$$

(8) $\iint_D xy\,dx\,dy$ $\quad$ D: $x^2+y^2 \le 4$ $\quad xy \ge 0$



$$x = r\cos\theta, \quad y = r\sin\theta$$

$$= 2\int_0^{\frac{\pi}{2}} d\theta \int_0^2 r^3 \sin\theta\cos\theta\,dr$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{2^4}{4} \sin\theta\cos\theta\,d\theta = 8\left[\frac{1}{2}\sin^2\theta\right]_0^{\frac{\pi}{2}} = 4$$

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(9) $\iint y^2 dx\, dy$ $\qquad$ D: $x^2+y^2 \leq 1$ $\quad x = r\cos\theta,\ y = r\sin\theta,\ J = r$

$$= \int_0^{2\pi} d\theta \int_0^1 r^3 \sin^2\theta\, dr = \frac{1}{4}\int_0^{2\pi} \sin^2\theta\, d\theta = \frac{1}{4}\cdot 4 \cdot \frac{\pi}{4} = \frac{\pi}{4}$$

(10) $\iint_D \left(\frac{x^2}{p} + \frac{y^2}{q}\right)^2 dx\, dy$ $\qquad$ D: $x^2+y^2 \leq a^2$, p, q const

$x = r\cos\theta,\ y = r\sin\theta$

$$= \int_0^{2\pi} d\theta \int_0^a \left(\frac{r^4}{p^2}\cos^4\theta + 2\frac{r^4}{pq}\cos^2\theta\sin^2\theta + \frac{r^4}{q^2}\sin^4\theta\right) r\, dr$$

$$= \frac{a^6}{6}\int_0^{2\pi}\left(\frac{1}{p^2}\cos^4\theta + \frac{2}{pq}(\cos^2\theta - \cos^4\theta) + \frac{1}{q^2}\sin^4\theta\right) d\theta$$

$$= \frac{2}{3}a^6\left(\frac{1}{p^2}\,\frac{3}{4}\,\frac{\pi}{4} + \frac{2}{pq}\,\frac{\pi}{4} - \frac{2}{pq}\,\frac{3}{4}\,\frac{\pi}{4} + \frac{1}{q^2}\,\frac{3}{4}\,\frac{\pi}{4}\right)$$

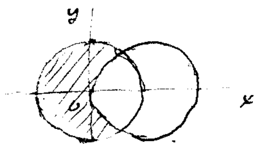
$$= \frac{2}{3}a^6\left(\frac{1}{p^2}\cdot\frac{3}{16} + \frac{2}{16}\,\frac{1}{pq} + \frac{1}{q^2}\,\frac{3}{16}\right)\pi$$

$$= \left(\frac{1}{8p^2} + \frac{1}{12pq} + \frac{1}{8q^2}\right)a^6\pi$$

(11) $\iint_D x^2 dx\, dy$ $\qquad$ D: $x^2+y^2 \leq 1$ $\quad x = r\cos\theta,\ y = r\sin\theta$

$$= \int_0^{2\pi} d\theta \int_0^1 r^3\cos^2\theta\, dr = \frac{1}{4}\int_0^{2\pi}\cos^2\theta\, d\theta = \frac{1}{4}\cdot 4\cdot\frac{\pi}{4} = \frac{\pi}{4}$$

1.5 (1) $\iint_D \sqrt{x^2+y^2}\, dx\, dy$ $\qquad$ D: $x^2+y^2 \leq a^2$, $\quad x^2+y^2-2ax \geq 0$



$$= 2\int_{\frac{\pi}{2}}^{\pi} d\theta \int_0^a r^2 dr + 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\theta \int_{2a\cos\theta}^{a} r^2 dr$$

$x = r\cos\theta,\ y = r\sin\theta$

$J = r$

$r \leq a$ $\quad r \geq 2a\cos\theta$

$a = 2a\cos\theta$ $\quad \cos\theta = \frac{\pi}{3}$

$$= \frac{2}{3}a^3\,\frac{\pi}{2} + 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{3}(a^3 - 8a^3\cos^3\theta)\, d\theta$$

$$= \frac{a^3\pi}{3} + \frac{a^3\pi}{9} - \frac{16}{3}a^3\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(1-\sin^2\theta)\cos\theta\, d\theta$$

$$= \frac{4a^3\pi}{9} - \frac{16}{3}a^3\left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \frac{4a^3}{9}\pi - \frac{16}{3}a^3\left(\frac{2-\sqrt{3}}{2} - \frac{1}{3}\,\frac{8-3\sqrt{3}}{8}\right)$$

$$= \frac{4}{9}a^3\pi - \frac{16}{3}a^3\cdot\frac{16-9\sqrt{3}}{24} = \frac{4}{9}a^3\pi - \frac{32-18\sqrt{3}}{9}a^3$$

$$= \frac{2}{9}a^3(2\pi - 16 + 9\sqrt{3})$$

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1.5 (2) $\iint_D \sqrt{\dfrac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$　　D : $x^2+y^2 \le r^2$,　$x \ge 0$, $y \ge 0$

(r=1のミスプリントミス)

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, r\,dr$$

$x = r\cos\theta$　$y = r\sin\theta$,　$J = r$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_1^0 t\left(\frac{-2t}{(1+t^2)^2}\right) dt$$

$\sqrt{\dfrac{1-r^2}{1+r^2}} = t$ とおく

$1-r^2 = (1+r^2)t^2$　　$r^2 = \dfrac{1-t^2}{1+t^2}$

$2r\,dr = \dfrac{-2t(1+t^2)-2t(1-t^2)}{(1+t^2)^2}\,dt = \dfrac{-4t}{(1+t^2)^2}\,dt$

$$= \frac{\pi}{2}\int_0^1 \frac{2t^2}{(1+t^2)^2}\,dt$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \frac{2\tan^2\theta}{\sec^4\theta}\sec^2\theta\,d\theta$$

$t = \tan\theta$　$dt = \sec^2\theta\,d\theta$

$$= \pi\int_0^{\frac{\pi}{4}} \sin^2\theta\,d\theta$$

$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$= \pi\int_0^{\frac{\pi}{4}} \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta = \pi\,\frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} = \frac{\pi}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

(3) $\iint_D \dfrac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy$　　D ; $x^2+y^2 \le 1$　　$x = r\cos\theta$, $y = r\sin\theta$

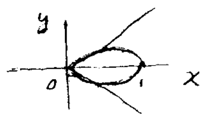
$$= \int_0^{2\pi} d\theta \int_0^1 \frac{r}{\sqrt{1-r^2}}\,dr = 2\pi\left[-\sqrt{1-r^2}\right]_0^1 = 2\pi$$

(4) $\iint_D (1+x^2+y^2)\,dx\,dy$　　D: $(x^2+y^2)^2 \le x^2-y^2$　　$x \ge 0$

$x = r\cos\theta$　$y = r\sin\theta$　　$J = r$

$r^4 \le r^2\cos 2\theta$

$r^2 \le \cos 2\theta$



$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \int_0^{\sqrt{\cos 2\theta}} (1+r^2)\,r\,dr$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{r^2}{2} + \frac{r^4}{4}\right]_0^{\sqrt{\cos 2\theta}} d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta\right) d\theta$$

$2\theta = t$

$d\theta = \frac{1}{2}\,dt$

$$= \left[\frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\ \frac{1}{2}\,dt$$

$$= \frac{2}{4} + \frac{1}{8}\cdot 2\cdot\frac{\pi}{4} = \frac{1}{2} + \frac{1}{16}\pi$$

(5) $\iint_D \dfrac{1}{\sqrt{1+x^2+y^2}}\,dx\,dy$　　D : $x^2+y^2 \le 1$　　$x = r\cos\theta$　$y = r\sin\theta$

$J = r$

$$= \int_0^{2\pi} d\theta \int_0^1 \frac{r}{\sqrt{1+r^2}}\,dr$$

$$= 2\pi\left[\sqrt{1+r^2}\right]_0^1 = 2\pi(\sqrt{2} - 1)$$

(6) $\iint_D \sqrt{x^2+y^2}\,dx\,dy$　　D : $(x-1)^2+y^2 \le 1$　　$x^2+y^2 \le 2x$

$x = r\cos\theta$, $y = r\sin\theta$　　$r \le 2\cos\theta$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} r^2\,dr = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8}{3}\cos^3\theta\,d\theta = \frac{8}{3}\cdot 2\cdot\frac{2}{3} = \frac{32}{9}$$

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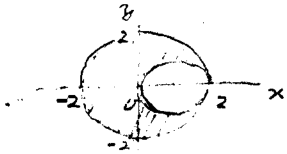
(7) $\iint_D \sqrt{x^2+y^2}\,dx\,dy$ D: $2x \le x^2+y^2 \le 4$



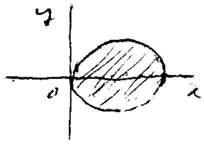
$$=2\left\{\int_{\frac{\pi}{2}}^{\pi} d\theta \int_0^2 r^2\,dr + \int_0^{\frac{\pi}{2}} d\theta \int_{2\cos\theta}^{2} r^2\,dr\right\}$$

$x=r\cos\theta$, $y=r\sin\theta$, $2\cos\theta \le r \le 2$, $J=r$

$$=2\left\{\int_{\frac{\pi}{2}}^{\pi}\frac{8}{3}\,d\theta + \int_0^{\frac{\pi}{2}}\left(\frac{8}{3}-\frac{8}{3}\cos^3\theta\right)d\theta\right\}$$

$$=2\left\{\frac{4}{3}\pi+\frac{4}{3}\pi-\frac{8}{3}\cdot\frac{2}{3}\right\}=\frac{16}{9}(3\pi-2)$$

(8) $\iint_D \sqrt{9-x^2-y^2}$ D: $x^2+y^2\le 3x$



$$=2\int_0^{\frac{\pi}{2}} d\theta \int_0^{3\cos\theta}\sqrt{9-r^2}\,r\,dr$$

$x=r\cos\theta$, $y=r\sin\theta$, $J=r$, $0\le r<3\cos\theta$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$$=2\int_0^{\frac{\pi}{2}}\left[-\frac{1}{3}\left(\sqrt{9-r^2}\right)^3\right]_0^{3\cos\theta}d\theta$$

$$=\frac{2}{3}\int_0^{\frac{\pi}{2}}(27-27\sin^3\theta)\,d\theta=\frac{2}{3}\left(27\frac{\pi}{2}-27\cdot\frac{2}{3}\right)=9\pi-12$$

(9) $\iint_D \frac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy$ D: $x^2+y^2\le 1$

$$=\int_0^{2\pi} d\theta \int_0^1 \frac{r}{\sqrt{1-r^2}}\,dr$$

$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$ $J=r$

$$=\int_0^{2\pi}\left[-\sqrt{1-r^2}\right]_0^1 d\theta=\int_0^{2\pi}1\,d\theta=2\pi$$

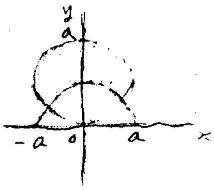
(10) $\iint_D xy\,dx\,dy$ D: $x^2+y^2\le a^2$ $x\ge 0$ $y\ge 0$

$$=\int_0^{\frac{\pi}{2}} d\theta \int_0^a r^3\cos\theta\sin\theta\,dr$$

$x=r\cos\theta$, $y=r\sin\theta$, $J=r$, $0\le r\le a$

$$=\int_0^{\frac{\pi}{2}}\frac{a^4}{4}\sin\theta\cos\theta\,d\theta=\frac{a^4}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}}=\frac{a^4}{8}$$

1.6 (1) $\iint_D |x|\,dx\,dy$ D: $a\le r\le 2a\sin\theta$



$$=\int_{\frac{\pi}{6}}^{\frac{5}{6}\pi} d\theta \int_a^{2a\sin\theta} |r\cos\theta|\,r\,dr$$

$\left[\begin{array}{l} x=r\cos\theta \\ y=r\sin\theta \end{array}\right.$, $a=2a\sin\theta$ $\theta=\frac{\pi}{6}$

$$=2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} d\theta \int_a^{2a\sin\theta} r^2\cos\theta\,d\theta=2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1}{3}(8a^3\sin^3\theta\cos\theta-a^3\cos\theta)\,d\theta$$

$$=\frac{2}{3}\left[8a^3\frac{1}{4}\sin^4\theta-a^3\sin\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}=\frac{2}{3}\left\{2a^3-a^3-\left(\frac{1}{8}a^3-\frac{a^3}{2}\right)\right\}$$

$$=\frac{11}{12}a^3$$

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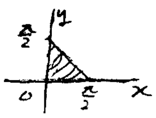
16 (2) $\iint_D |xy|\,dx\,dy$ $\qquad$ $D:\ 2 \le \sqrt{x^2+y^2} \le 3$

$x = r\cos\theta$, $y = r\sin\theta$ $\qquad J = r \qquad 2 \le r \le 3$

$= \int_0^{2\pi} d\theta \int_2^3 r^2 |\sin\theta\cos\theta|\, r\,dr$

$= 4\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\,d\theta \int_2^3 r^3\,dr = 4\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_2^3 = 2\cdot\frac{81-16}{4} = \frac{65}{2}$

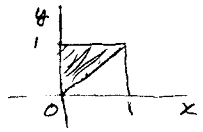
(3) $\iint_D \sin(x+y)\,dx\,dy$ $\qquad$ $D:\ x \ge 0,\ y \ge 0 \quad x+y \le \frac{\pi}{2}$



$= \int_0^{\frac{\pi}{2}} dx \int_0^{\frac{\pi}{2}-x} \sin(x+y)\,dy = \int_0^{\frac{\pi}{2}} \left[-\cos(x+y)\right]_0^{\frac{\pi}{2}-x} dx$

$= \int_0^{\frac{\pi}{2}} (\cos x - \cos\frac{\pi}{2})\,dx = \left[\sin x\right]_0^{\frac{\pi}{2}} = 1$

(4) $\iint_D \cos x\,dx\,dy$ $\qquad$ $D:\ 0 \le x \le y \le 1$



$= \int_0^1 dx \int_x^1 \cos x\,dy = \int_0^1 \left[y\cos x\right]_x^1 dx = \int_0^1 (1-x)\cos x\,dx$

$= \left[\sin x - x\sin x - \cos x\right]_0^1 = 1 - \cos 1$

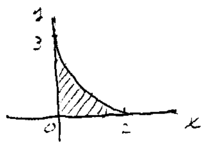
1.7 $\iint_D (x^2+y^2)\,dx\,dy$ $\qquad$ $D:\ \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$

$x = ar\cos\theta$, $y = br\sin\theta$ $\qquad J = abr \qquad 0 \le r \le 1$

$= \int_0^{2\pi} d\theta \int_0^1 ab(a^2\cos^2\theta + b^2\sin^2\theta)\,r^3\,dr$

$= \frac{ab}{4}\int_0^{2\pi} (a^2\cos^2\theta + b^2\sin^2\theta)\,d\theta = \frac{ab}{4}(a^2\pi + b^2\pi) = \frac{ab\pi}{4}(a^2+b^2)$

1.8 (1) $\iint_D y\,dx\,dy$ $\qquad$ $D:\ x \ge 0,\ y \ge 0 \quad \sqrt{\frac{x}{2}} + \sqrt{\frac{y}{3}} \le 1$



$\frac{y}{3} = (1 - \frac{\sqrt{x}}{\sqrt{2}})^2 = (1 - \sqrt{2x} + \frac{x}{2})$

$y = 3 + \frac{3}{2}x - 3\sqrt{2x}$

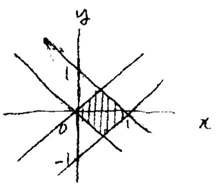
$= \int_0^2 dx \int_0^{3+\frac{3}{2}x-3\sqrt{2x}} y\,dy$

$= \int_0^2 \left[\frac{1}{2}y^2\right]_0^{3+\frac{3}{2}x-3\sqrt{2x}} dx = \frac{1}{2}\int_0^2 (\frac{9}{4}x^2 + 27x - 9\sqrt{2}\,x^{\frac{3}{2}} - 18\sqrt{2}\,x^{\frac{1}{2}} + 9)\,dx$

$= \frac{1}{2}\left[\frac{3}{4}x^3 + \frac{27}{2}x^2 - \frac{18\sqrt{2}}{5}x^{\frac{5}{2}} - 12\sqrt{2}\,x^{\frac{3}{2}} + 9x\right]_0^2$

$= \frac{1}{2}(6 + 54 - \frac{18\times 8}{5} - 48 + 18) = \frac{3}{5}$

(2) $\iint_D x\,dx\,dy$ $\qquad$ $D:\ 0 \le x+y \le 1,\ 0 \le x-y \le 1$



$u = x+y$ $\qquad x = \frac{1}{2}(u+v)$

$v = x-y$ $\qquad y = \frac{1}{2}(u-v)$

$|J| = \frac{1}{2}$

$= \int_0^1 du \int_0^1 \frac{1}{4}(u+v)\,dv$

$= \frac{1}{4}\int_0^1 (u+\frac{1}{2})\,du = \frac{1}{4}$

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(3) $\iint_D \{(x-y)^2+(x+2y)^2\}\,dx\,dy$

$$= \int_{-2}^{2} du \int_{-1}^{1} (u^2+v^2)\frac{1}{3}\,du\,dv$$

$$= \frac{1}{3}\int_{-2}^{2}\Big[u^2v+\frac{v^3}{3}\Big]_{-1}^{1} = \frac{1}{3}\int_{-2}^{2}\Big(2u^2+\frac{2}{3}\Big)du$$

$$= \frac{1}{3}\Big[\frac{2}{3}u^3+\frac{2}{3}u\Big]_{-2}^{2}$$

$$= \frac{1}{3}\Big(\frac{32}{3}+\frac{8}{3}\Big) = \frac{40}{9}$$

D : $|x-y| \leqq 2$ $|x+2y| \leqq 1$.

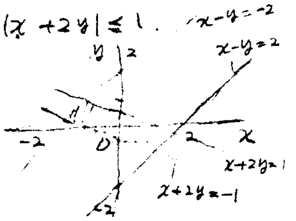
$u = x-y$

$v = x+2y$

$x = \frac{2}{3}u+\frac{1}{3}v$

$y = -\frac{1}{3}u+\frac{1}{3}v$

$$J = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}$$



(4) $\iint_D (x+y)e^{x-y}\,dx\,dy$

$$= \int_0^1 du \int_0^1 u e^v \frac{1}{2}\,dv$$

$$= \frac{1}{2}\int_0^1 u(e-1)\,du = \frac{e-1}{4}$$

D : $0 \leqq x+y \leqq 1$

: $0 \leqq x-y \leqq 1$

$u = x+y$ $\quad x = \frac{1}{2}(u+v)$

$v = x-y$ $\quad y = \frac{1}{2}(u-v)$

$|J| = \frac{1}{2}$

1.9 $\iiint_D (x+y+z)\,dx\,dy\,dz$

D : $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leqq 1$

$x = au$ $\quad y = bv$ $\quad z = cw$ とおくと

$J = abc$ $\quad u^2+v^2+w^2 \leqq 1$

D' ; $u^2+v^2+w^2 \leqq 1$

$$= abc\iiint_{D'} (au+bv+cw)\,du\,dv\,dw$$

$$= abc\iint_{u^2+v^2\leqq 1} du\,dv \int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-w^2}} \{(au+bv)+cw\}\,dw$$

$$= abc\iint_{u^2+v^2\leqq 1} \Big[(au+bv)w+\frac{c}{2}w^2\Big]_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}} du\,dv$$

$$= 2abc\iint_{u^2+v^2\leqq 1} (au+bv)\sqrt{1-u^2-v^2}\,du\,dv$$

$\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$ $\quad J = r$

$0 \leqq r \leqq 1$

$$= 2abc\int_0^{2\pi} (a\cos\theta+b\sin\theta)\,d\theta \int_0^1 \sqrt{1-r^2}\,r^2\,dr$$

$$= 0$$

P.62

§2 重積分

2.1 $z = \log\frac{1}{r} = -\log r \qquad r = \sqrt{x^2+y^2}$

$= -\frac{1}{2}\log(x^2+y^2)$

(1) $z_x = \frac{-x}{x^2+y^2} \qquad z_{xx} = \frac{-(x^2+y^2)+2x^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$z_y = \frac{-y}{x^2+y^2} \qquad z_{yy} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\therefore z_{xx}+z_{yy} = 0$

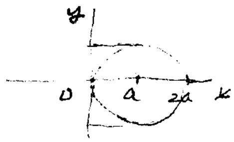
(2) $\int_A z\,dx\,dy \qquad A = \{(x,y) \mid x^2+y^2 \leqq R^2,\ R>0\}$ （$\leqq R^2$ ミスプリ）

$= \iint_A -\frac{1}{2}\log(x^2+y^2)\,dx\,dy \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad r \leqq R,\ 0 \leqq \theta \leqq 2\pi \qquad J = r$

$= -\int_0^{2\pi} d\theta \int_0^R \log r \cdot r\,dr$

$= -\int_0^{2\pi} \left[\frac{r^2}{2}\log r - \frac{r^2}{4}\right]_0^R d\theta = \int_0^{2\pi}\left(\frac{R^2}{4} - \frac{R^2}{2}\log_e R\right)d\theta = \frac{R^2\pi}{2} - \pi R^2\log R$

2.2 $\iint_D |xy|\,dx\,dy \qquad D:\ x^2+y^2 \leqq 2ax$



$x = r\cos\theta$

$y = r\sin\theta$

$r \leqq 2a\cos\theta$

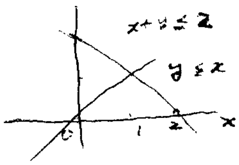
$-\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2} \qquad J = r$

$= 2\int_0^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} r^3\sin\theta\cos\theta\,dr$

$= 2\int_0^{\frac{\pi}{2}} \frac{16}{4}a^4\cos^5\theta\sin\theta\,d\theta$

$= 8a^4\left[-\frac{\cos^6\theta}{6}\right]_0^{\frac{\pi}{2}} = \frac{4}{3}a^4$

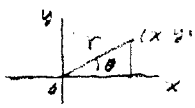
2.3 $\iint_D 2xy\,dx\,dy \qquad D:\ y \geqq 0,\ y \leqq x,\ x+y \leqq 2$



$= \int_0^1 dy \int_y^{2-y} 2xy\,dx = \int_0^1 \left[x^2y\right]_y^{2-y} dy$

$= \int_0^1 ((2-y)^2y - y^3)\,dy = \left[\frac{y^4}{4} - \frac{4y^3}{3} + \frac{4}{2}y^2 - \frac{y^4}{4}\right]_0^1 = \frac{2}{3}$

2.4 (1) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$



$|J| = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \left|\begin{matrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{matrix}\right| = r$

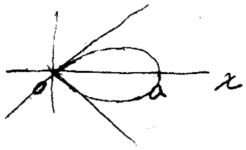
(2) $\iint_D \sqrt{a^2-x^2-y^2}\,dx\,dy \qquad D:\ x^2+y^2 \leqq a^2$

$= \int_0^{2\pi} d\theta \int_0^a \sqrt{a^2-r^2}\,r\,dr = \int_0^{2\pi}\left[-\frac{1}{3}(a^2-r^2)^{\frac{3}{2}}\right]_0^a d\theta = \frac{a^3}{3}\int_0^{2\pi} d\theta = \frac{2a^3}{3}\pi$

P62, P63

2.5, 2.3に同じ

2.6 (1) $r = a\cos 2\theta \qquad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$



(2) $\iint_D x\,dx\,dy \qquad x = r\cos\theta \quad y = r\sin\theta$

$J = r$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \int_0^{a\cos 2\theta} r^2 \cos\theta \, d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{3} a^3 \cos^3 2\theta \cos\theta \, d\theta = \frac{a^3}{3} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 - 2\sin^2\theta)^3 \cos\theta \, d\theta$$

$$= \frac{a^3}{3} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} -(8\sin^6\theta - 12\sin^4\theta + 6\sin^2\theta - 1)\cos\theta \, d\theta$$

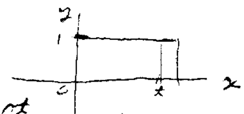
$$= -\frac{a^3}{3}\left[\frac{8}{7}\sin^7\theta - \frac{12}{5}\sin^5\theta + 2\sin^3\theta - \sin\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= -\frac{2a^3}{3}\left(\frac{1}{7\sqrt{2}} - \frac{3}{5\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = \frac{-2a^3}{3\sqrt{2}}\left(\frac{1}{7} - \frac{3}{5}\right)$$

$$= \frac{2a^3}{3\sqrt{2}} \frac{16}{35} = \frac{16\sqrt{2}}{105} a^3$$

2.7 $F(t) = \iint_{I_t} f(x,y)\,dx\,dy \qquad I_t = \{(x,y) \mid 0 \le x \le t,\ 0 \le y \le 1\}$

$$= \int_0^1 dy \int_0^t f(x,y)\,dx$$



$$F(t+\Delta t) - F(t) = \int_0^1 \left\{\int_0^{t+\Delta t} f(x,y)\,dx - \int_0^t f(x,y)\,dx\right\} dy$$

$$= \int_0^1 \left\{\int_t^{t+\Delta t} f(x,y)\,dx\right\} dy = \int_t^{t+\Delta t} \left\{\int_0^1 f(x,y)\,dy\right\} dx$$

$$\therefore F'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\{F(t+\Delta t) - F(t)\} = \int_0^1 f(x,y)\,dy$$

2.8 (1) $D: \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \qquad u = \frac{x}{a},\ v = \frac{y}{b} \qquad x = au,\ y = bv \qquad J = ab$

$\iint_D f(x,y)\,dx\,dy \qquad D': u^2 + v^2 \le 1$

$$= \iint_{D'} ab f(au, bv)\,du\,dv$$

(2) $\iint_D (x^2+y^2)\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}\,dx\,dy = \iint_{D'} ab(a^2u^2 + b^2v^2)\sqrt{u^2+v^2}\,du\,dv$

$u = r\cos\theta \quad v = r\sin\theta \qquad J = r$

$$= \int_0^{2\pi} d\theta \int_0^1 ab(a^2\cos^2\theta + b^2\sin^2\theta) r^4\,dr$$

$$= \frac{ab}{5}\int_0^{2\pi} (a^2\cos^2\theta + b^2\sin^2\theta)\,d\theta = \frac{ab\pi}{5}(a^2+b^2)$$

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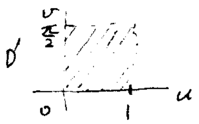
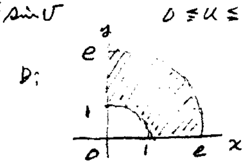
2.9 (1) D : $x = e^u \cos v \quad y = e^u \sin v \qquad 0 \leq u \leq 1 \quad 0 \leq v \leq \frac{\pi}{2}$





$e^{2u} = x^2 + y^2$

$\tan v = \frac{y}{x}$

$u = \frac{1}{2} \log (x^2+y^2)$

$v = \tan^{-1} \frac{y}{x}$

(2) $J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} e^u \cos v, & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix} = e^{2u}$

(3) $\iint_D xy\,dx\,dy = \iint_{D'} e^{2u} \cos v \sin v\, e^{2u}\, du\,dv$

$= \int_0^{\frac{\pi}{2}} \cos v \sin v\, dv \int_0^1 e^{4u}\, du = \frac{1}{2}\left[\sin^2 v\right]_0^{\frac{\pi}{2}} \cdot \left[\frac{1}{4} e^{4u}\right]_0^1$

$= \frac{1}{8}(e^4 - 1)$

2.10 $I = \iint_D \frac{1-x^2-y^2}{1+x^2+y^2}\,dx\,dy \qquad$ D : $x^2+y^2 \leq 1, \quad x \geq 0, y \geq 0$

(1) $x = r\cos\theta \quad J = r. \qquad D' : 0 \leq r \leq 1, \quad 0 \leq \theta \leq \frac{\pi}{2}$

$y = r \sin\theta$

$I = \iint_{D'} \frac{1-r^2}{1+r^2} r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} d\theta \int_0^1 \left(-1 + \frac{2}{1+r^2}\right) r\,dr$

(2) $I = \int_0^{\frac{\pi}{2}} \left[-\frac{r^2}{2} + \log(1+r^2)\right]_0^1 d\theta = \int_0^{\frac{\pi}{2}} \left(-\frac{1}{2} + \log 2\right) d\theta$

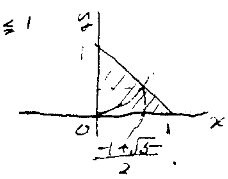
$= \frac{\pi}{2}\left(\log 2 - \frac{1}{2}\right)$

2.11 $I = \iint_D \max(x^2, y)\,dx\,dy \qquad$ D : $0 \leq x,\ 0 \leq y,\ x + y \leq 1$

$x + x^2 = 1$

$x^2 + x - 1 = 0$

$x = \frac{1}{2}(-1+\sqrt{5})$



$= \int_0^{\frac{-1+\sqrt{5}}{2}} dx \int_{x^2}^{1-x} y\,dy + \int_0^{\frac{-1+\sqrt{5}}{2}} dx \int_0^{x^2} x^2\,dy$

$+ \int_{\frac{-1+\sqrt{5}}{2}}^1 dx \int_0^{1-x} x^2\,dy$

$= \int_0^{\frac{-1+\sqrt{5}}{2}} \frac{1}{2}\{(1-x)^2 - x^4\}\,dx + \int_0^{\frac{-1+\sqrt{5}}{2}} x^4\,dx + \int_{\frac{-1+\sqrt{5}}{2}}^1 x^2(1-x)\,dx$

$= \frac{1}{2}\left[-\frac{(1-x)^3}{3} - \frac{x^5}{5}\right]_0^{\frac{-1+\sqrt{5}}{2}} + \left[\frac{x^5}{5}\right]_0^{\frac{-1+\sqrt{5}}{2}} + \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_{\frac{-1+\sqrt{5}}{2}}^1$

$= (25\sqrt{5} - 31)/120$

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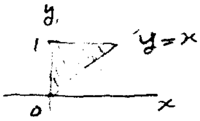
12, $\iint_D (x^2 - y)\,dx\,dy$ $\quad 0 \le x \le y \le 1$

$= \int_0^1 dx \int_x^1 (x^2 - y)\,dy = \int_0^1 \left[x^2 y - \frac{y^2}{2}\right]_x^1 dx = \int_0^1 \left(x^2 - \frac{1}{2} - x^3 + \frac{x^2}{2}\right) dx$

$= \frac{1}{3} - \frac{1}{2} - \frac{1}{4} + \frac{1}{6} = -\frac{1}{4}$

2.13 $\iint_D \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$ $\quad D:$



$= \int_0^1 dy \int_0^y \frac{x}{\sqrt{x^2+y^2}}\,dx$

$= \int_0^1 \left[\sqrt{x^2+y^2}\right]_0^y dy = \int_0^1 (\sqrt{2}\,y - y)\,dy = \frac{\sqrt{2}-1}{2}\left[y^2\right]_0^1 = \frac{\sqrt{2}-1}{2}$

2.14 $\iint_D (x^2+y^2)\,dx\,dy$ $\quad D: x \ge 0 \quad y \ge 0 \quad x+y \le 1$

$= \int_0^1 dx \int_0^{1-x} (x^2+y^2)\,dy = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx$

$= \int_0^1 \left\{x^2(1-x) + \frac{1}{3}(1-x)^3\right\} dx = \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{1}{12}(1-x)^4\right]_0^1 = \frac{1}{3} - \frac{1}{4} + \frac{1}{12} = \frac{1}{6}$

2.15 $\iint_D (x+y)\,e^{x-y}\,dx\,dy$ $\quad D: 0 \le x+y \le 1 \quad 0 \le x-y \le 1$

$u = x+y \quad x = \frac{1}{2}(u+v) \quad J = \frac{1}{2}$

$v = x-y \quad y = \frac{1}{2}(u-v)$

$= \frac{1}{2}\int_0^1 u\,du \int_0^1 e^v\,dv$

$= \frac{1}{4}(e-1)$

2.16 $\iint_D \frac{\log(x^2+y^2)}{x^2+y^2}\,dx\,dy$ $\quad D: x \ge 0, y \ge 0 \quad 1 \le x^2+y^2 \le 4$

$x = r\cos\theta \quad J = r \quad 1 \le r \le 2$

$y = r\sin\theta \quad 0 \le \theta \le \frac{\pi}{2}$

$= \int_0^{\frac{\pi}{2}} d\theta \int_1^2 \frac{2\log r}{r^2}\, r\,dr$

$= \frac{\pi}{2}\left[(\log r)^2\right]_1^2 = \frac{\pi}{2}(\log 2)^2$

2.17 $\iint_D x^2 \exp(x^2+y^2)\,dx\,dy$ $\quad D: x^2+y^2 \le 4 \quad x \ge 0$

$x = r\cos\theta \quad J = r \quad 0 \le r \le 2$

$y = r\sin\theta \quad 0 \le \theta \le \pi$

$= \int_0^\pi \cos^2\theta\,d\theta \int_0^2 r^3 e^{r^2}\,dr$

$= \frac{\pi}{2}\left\{\left[\frac{r^2 e^{r^2}}{2}\right]_0^2 - \int_0^2 r e^{r^2}\,dr\right\}$

$= \frac{\pi}{2}\left\{2e^4 - \frac{1}{2}\left[e^{r^2}\right]_0^2\right\} = \frac{3}{4}(e^4 - 1)$

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§ 3 広義積分

3.1 (1) $\iint_D \frac{x+y}{x^2+y^2}\,dx\,dy$ D: $0 \le x \le 1$ $0 \le y \le x$

$= \int_0^1 dx \int_0^x \left(\frac{x}{x^2+y^2} + \frac{y}{x^2+y^2}\right) dy = \int_0^1 \left[\tan^{-1}\frac{y}{x} + \frac{1}{2}\log(x^2+y^2)\right]_0^x dx$

$= \int_0^1 \left\{\frac{\pi}{4} + \frac{1}{2}(\log 2x^2 - \log x^2)\right\} dx = \frac{\pi}{4} + \frac{1}{2}\int_0^1 (\log 2)\,dx = \frac{\pi}{4} + \frac{1}{2}\log 2$

(2) $\iint \frac{1}{(x+y)^{\frac{3}{2}}}\,dx\,dy$ $1/(x+y)^{-\frac{3}{2}} \to (x+y)^{-\frac{3}{2}}$ プリントミス

D: $0 \le x \le 1$ $0 \le y \le 1$

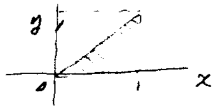
$= \int_0^1 dx \int_0^1 (x+y)^{-\frac{3}{2}} dy$

$= \int_0^1 \left[-2(x+y)^{-\frac{1}{2}}\right]_0^1 dx = \int_0^1 \left\{2x^{-\frac{1}{2}} - 2(x+1)^{-\frac{1}{2}}\right\} dx = \left[4x^{\frac{1}{2}} - 4(x+1)^{\frac{1}{2}}\right]_0^1$

$= 4 - 4\sqrt{2} + 4 = 8 - 4\sqrt{2}$

(3) $\iint_D \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$ D: $0 \le x \le 1$, $0 \le y \le x$

$= \int_0^1 dy \int_y^1 \frac{x}{\sqrt{x^2+y^2}}\,dx = \int_0^1 \left[\sqrt{x^2+y^2}\right]_y^1 dy$



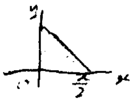
$= \int_0^1 \left(\sqrt{1+y^2} - \sqrt{2y^2}\right) dy$

$= -\sqrt{2}\left[\frac{1}{2}y^2\right]_0^1 + \frac{1}{2}\left[y\sqrt{y^2+1} + \log(y+\sqrt{y^2+1})\right]_0^1$

$= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{1}{2}\log(1+\sqrt{2}) = \frac{1}{2}\log(1+\sqrt{2})$

3.2 (1) $\iint_D \sin^{-1}\frac{2xy}{x^2+y^2}\,dx\,dy$ D: $0 \le x$, $0 \le y$, $0 \le x+y \le \frac{\pi}{2}$

$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\pi/2\sqrt{2}\sin(\theta+\frac{\pi}{4})} 2\theta r\,dr$



$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $J = r$ $0 \le \theta \le \frac{\pi}{2}$

$= \int_0^{\frac{\pi}{2}} 2\theta \left[\frac{r^2}{2}\right]_0^{\frac{\pi}{2\sqrt{2}\sin(\theta+\frac{\pi}{4})}} d\theta$

$\frac{2xy}{x^2+y^2} = 2\cos\theta\sin\theta = \sin 2\theta$

$= \int_0^{\frac{\pi}{2}} \theta \frac{\pi^2}{8\sin^2(\theta+\frac{\pi}{4})}\,d\theta$

$\therefore \sin^{-1}\frac{2xy}{x^2+y^2} = 2\theta$

$= \frac{\pi^2}{8}\left\{\left[\theta\left(-\cot(\theta+\frac{\pi}{4})\right)\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cot(\theta+\frac{\pi}{4})\,d\theta\right\}$

$0 \le r(\cos\theta + \sin\theta) \le \frac{\pi}{2}$

$\sqrt{2}\,r\sin(\theta+\frac{\pi}{4}) \le \frac{\pi}{2}$

$= \frac{\pi^2}{8}\left\{\frac{\pi}{2} + \left[\log|\sin(\theta+\frac{\pi}{4})|\right]_0^{\frac{\pi}{2}}\right\}$

$r \le \frac{\pi}{2\sqrt{2}\sin(\theta+\frac{\pi}{4})}$

$= \frac{\pi^2}{8}\left\{\frac{\pi}{2} + \log\left|\frac{\sin\frac{3\pi}{4}}{\sin\frac{\pi}{4}}\right|\right\} = \frac{\pi^3}{16}$

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(2) $\iint_D \cos\frac{x-y}{x+y}\,dx\,dy$ D: $x \geq 0,\ y \geq 0,\ 0 \leq x+y \leq 1$

$= \int_0^1 du \int_{-u}^{u} \cos\frac{v}{u}\,dv$ $u = x+y$

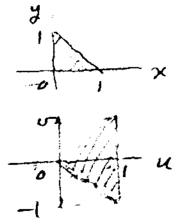
$v = x-y$

$= \int_0^1 \left[u \sin\frac{v}{u}\right]_{-u}^{u} du$ $x = \frac{1}{2}(u+v)$

$y = \frac{1}{2}(u-v)$

$= 2\int_0^1 u \sin 1\,du = 2\left[\frac{u^2}{2}\right]_0^1 \sin 1$ $|J| = \frac{1}{2}$

$= \sin 1$



3.3 (1) $\iint_D e^{-(x^2+y^2)}\,dx\,dy$ D: $x \geq 0 \quad y \geq 0$

$x = r\cos\theta,\ y = r\sin\theta$

$0 \leq R,\ 0 \leq \theta \leq \frac{\pi}{2}$ $J = r$

$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} r e^{-r^2} dr$

$= \int_0^{\frac{\pi}{2}} \left[-\frac{1}{2} e^{-r^2}\right]_0^{\infty} d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{4}$

(2) $\iint_D \frac{1}{(x^2+y^2)^2}\,dx\,dy$ D; $x^2+y^2 \geq 1$

$x = r\cos\theta,\ y = r\sin\theta$ $J = r$

$= \int_0^{2\pi} d\theta \int_1^{\infty} \frac{1}{r^4} r\,dr$ $r \geq 1,\ 0 \leq \theta \leq 2\pi$

$= \int_0^{2\pi} \left[-\frac{1}{2r^2}\right]_1^{\infty} d\theta = \frac{1}{2}\int_0^{2\pi} d\theta = \pi$

3.4 $\iint_D \frac{1}{(x^2+y^2)^p}\,dx\,dy$ D: $x^2+y^2 \leq a^2$

$x = r\cos\theta \quad y = r\sin\theta \quad J = r$

$0 \leq r \leq a \quad 0 \leq \theta \leq 2\pi$

$= \int_0^{2\pi} d\theta \int_0^{a} \frac{1}{r^{2p}} r\,dr$

$= \int_0^{2\pi} \left[\frac{1}{2-2p} r^{2-2p}\right]_0^{a} d\theta$ ∴ $2-2p>0$ のとき存在する

$0<p<1$

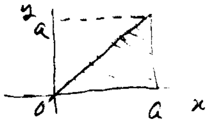
$= \frac{2\pi}{2(1-p)} a^{2(1-p)} = \frac{\pi}{1-p} a^{2(1-p)}$ $p>1$ のとき存在しない

3.5 (1) $\int \sin^{-1}\frac{x}{a}\,dx = x\sin^{-1}\frac{x}{a} - \int x \frac{1}{\sqrt{a^2-x^2}}\,dx = x\sin^{-1}\frac{x}{a} + \sqrt{a^2-x^2}$

(2) $\int_0^a dx \int_0^x \frac{f'(y)}{\sqrt{(a-x)(x-y)}}\,dy$ $\sqrt{\frac{x-y}{a-x}} = t$

$x = \frac{at^2+y}{1+t^2}$

$= \int_0^a dy \int_y^a \frac{f'(y)}{a-x} \frac{1}{\sqrt{\frac{x-y}{a-x}}}\,dx$ $\frac{dx}{dt} = \frac{(a-y)t}{(1+t^2)^2}$



$= \int_0^a dy \int_0^{\infty} \frac{f'(y)}{\frac{a-y}{1+t^2}} \frac{1}{t} \frac{(a-y)t}{(1+t^2)^2}\,dt = \int_0^a dy \int_0^{\infty} \frac{f'(y)}{1+t^2}\,dt = \int_0^a \left[\tan^{-1} t\right]_0^{\infty} f'(y)\,dy$

$= \frac{\pi}{2}\left[f(y)\right]_0^a = \frac{\pi}{2}\left(f(a) - f(0)\right)$

P.64 §.4 $\int_0^\infty \exp(-x^2)\,dx$

4.1 (1) $\int_0^\infty x^2 e^{-x^2}dx = [x(-\frac{1}{2}e^{-x^2})]_0^\infty + \int_0^\infty \frac{1}{2}e^{-x^2}dx = \frac{1}{2}\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4}$

(2) $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy = \int_0^{\frac{\pi}{2}}d\theta\int_0^\infty re^{-r^2}dr$ $\quad x=r\cos\theta,\ y=r\sin\theta \quad J=r$

$= \frac{\pi}{2}[-\frac{1}{2}e^{-r^2}]_0^\infty = \frac{\pi}{4}$

(3) $I=\int_0^\infty e^{-x^2}dx$ とおくと

$I^2 = \int_0^\infty\int_0^\infty e^{-x^2}\cdot e^{-y^2}dx\,dy = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy = \frac{\pi}{4}$

$\therefore\ I = \frac{\sqrt{\pi}}{2}$

(4) $\int_{-\infty}^\infty e^{-x^2}dx = 2\int_0^\infty e^{-x^2}dx = \sqrt{\pi}$

(5) $\int_{-\infty}^\infty\int_{-\infty}^\infty (x^2+y^2)e^{-(x^2+y^2)}dx\,dy$ $\quad x=r\cos\theta,\ y=r\sin\theta \quad J=r$

$= \int_0^{2\pi}d\theta\int_0^\infty r^3e^{-r^2}dr = 2\pi\{[-\frac{r^2}{2}e^{-r^2}]_0^\infty + \int_0^\infty re^{-r^2}dr\}$

$= 2\pi[-\frac{1}{2}e^{-r^2}]_0^\infty = \pi$

4.2 (1) $\int_0^\infty x^2e^{-x^2}dx = \frac{\sqrt{\pi}}{4}$ 4.1 (1) に同じ

(2) $I=\int_0^\infty e^{-x^2}dx$ とおくと $I^2 = \int_0^\infty\int_0^\infty e^{-x^2}e^{-y^2}dx\,dy = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy$

$= \int_0^{\frac{\pi}{2}}d\theta\int_0^\infty e^{-r^2}r\,dr = \frac{\pi}{2}[-\frac{1}{2}e^{-r^2}]_0^\infty = \frac{\pi}{4}$ $\quad x=r\cos\theta,\ y=r\sin\theta$ とおくと $J=r$

$I = \frac{\sqrt{\pi}}{2}$

4.3 (1) $\mathrm{Erf}\,x = \int_0^x e^{-t^2}dt = \int_0^x \sum_{n=0}^\infty \frac{(-1)^n}{n!}t^{2n}dt$

$= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)n!}x^{2n+1}$ 収束半径 $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{(2n+3)(n+1)!}{(2n+1)n!} = \infty$

$\therefore$ 収束半径 ∞

(2) $\mathrm{Erf}(1) = 1 - \frac{1}{3} + \frac{1}{5\cdot 2} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!}$

$= 1 - 0.3333 + 0.1 - 0.023 \fallingdotseq 0.7$

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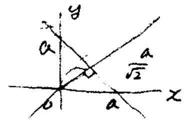
4.4 $z=\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$ $\qquad x+y\leq a$

(1) $V(a)=\frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{x^2+y^2}{2}}dx\,dy-\iint_D e^{-\frac{x^2+y^2}{2}}dx\,dy\right\}$ $\qquad$ D: $x+y\geq a$

(2) $V(a)=\frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}dx\int_{-\infty}^{\infty}e^{-\frac{y^2}{2}}dy-\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}dx\int_{\frac{a}{\sqrt{2}}}^{\infty}e^{-\frac{y^2}{2}}dy\right\}$



$=\frac{1}{2\pi}\left\{2\sqrt{\pi}\sqrt{\pi}-\sqrt{2\pi}\int_{\frac{a}{\sqrt{2}}}^{\infty}e^{-\frac{y^2}{2}}dy\right.$ $\qquad \frac{x}{\sqrt{2}}=t,\ dx=\sqrt{2}\,dt$

$=1+\frac{1}{\sqrt{2\pi}}\int_{\infty}^{\frac{a}{\sqrt{2}}}e^{-\frac{y^2}{2}}dy=1+\frac{1}{\sqrt{\pi}}\int_{\infty}^{\frac{a}{2}}e^{-t^2}dt$ $\qquad \frac{y}{\sqrt{2}}=t$

$\therefore \frac{dV(a)}{da}=\frac{1}{2\sqrt{\pi}}e^{-\frac{a^2}{4}}$

4.5 $\iint_D e^{-(x^2+y^2)}dx\,dy$ $\qquad$ D: $x\geq 0.\ y\geq 0$

$x=r\cos\theta.\ y=r\sin\theta \quad J=r$

$=\int_0^{\frac{\pi}{2}}d\theta\int_0^{\infty}re^{-r^2}dr$

$=\frac{\pi}{2}\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty}=\frac{\pi}{4}$

(1) $I(R)=\iint_D e^{-(x^2+y^2)}dS$ $\qquad$ D; $x\geq 0,\ y\geq 0 \quad x^2+y^2\leq R$

$=\int_0^{\frac{\pi}{2}}d\theta\int_0^{R}re^{-r^2}dr$ $\qquad x=r\cos\theta,\ y=r\sin\theta \quad J=r$

$=\frac{\pi}{2}\left[-\frac{1}{2}e^{-r^2}\right]_0^{R}$ $\qquad 0\leq r\leq R,\ 0\leq\theta\leq\frac{\pi}{2}$

$=\frac{\pi}{4}(1-e^{-R^2})$

(2) $I(R)<\int_0^a\int_0^a e^{-(x^2+y^2)}dx\,dy<I(\sqrt{2}R)$

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§5. 重積分の応用（体積）

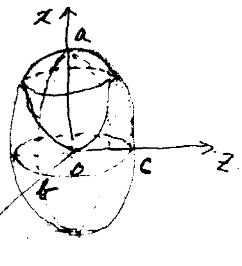
5.1 (1) $\iint_D \tan^{-1}\frac{y}{x}\,dx\,dy$ $\quad x=0,\ y=0,\ x^2+y^2=1,\ z=1$

$x=r\cos\theta$, $y=r\sin\theta$ $\quad J=r$ $\quad \frac{y}{x}=\tan\theta$

$$=\int_0^{\frac{\pi}{2}}d\theta\int_0^1\theta r\,dr=\int_0^{\frac{\pi}{2}}\frac{1}{2}\theta\,d\theta=\frac{\pi^2}{16}$$

(2) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{x}{a}$. $\quad a>0,\ b>0,\ c>0$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ だ円体，$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{x}{a}$. 錐面



$\frac{x^2}{a^2}+\frac{x}{a}=1$ $\quad x^2+ax-a^2=0$ $\quad x=\frac{-1\pm\sqrt{5}}{2}a$

$x>0$ より $x=\frac{\sqrt{5}-1}{2}a$

$\therefore\ \frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{\sqrt{5}-1}{2}$ $\quad x=a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}$

$$V=\iint_D\left\{a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}-a\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\right\}dy\,dz$$

$D:\ \frac{y^2}{b^2}+\frac{z^2}{c^2}\leqq\frac{\sqrt{5}-1}{2}$

$\begin{cases} y=br\cos\theta \\ z=cr\sin\theta \end{cases}$ とおく

$J=bcr$ $\quad 0\leqq r\leqq 1$

$\alpha=\sqrt{\frac{\sqrt{5}-1}{2}}$ とおく

$$=\int_0^{2\pi}d\theta\int_0^{\alpha}a(\sqrt{1-r^2}-r^2)bcr\,dr$$

$$=abc\cdot[\theta]_0^{2\pi}\left[-\frac{1}{3}(1-r^2)^{\frac{3}{2}}-\frac{r^4}{4}\right]_0^{\alpha}$$

$$=2abc\pi\ \frac{15-5\sqrt{5}}{24}=\frac{5}{12}(3-\sqrt{5})abc\pi$$

(3) $x^2+z^2=a^2$, $y^2+z^2=a^2$ $\quad x\geqq 0,\ y\geqq 0,\ z\geqq 0$

$$V=\iint_D\sqrt{a^2-z^2}\,dx\,dz$$

$D:\ x^2+z^2\leqq a^2\quad x\geqq 0,\ z\geqq 0$

$$=\int_0^a dx\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2-z^2}\,dz=\int_0^a\frac{1}{2}\left[z\sqrt{a^2-z^2}+a^2\sin^{-1}\frac{z}{a}\right]_0^{\sqrt{a^2-x^2}}dx$$

$$=\frac{1}{2}\int_0^a\left\{\sqrt{a^2-x^2}\,x+a^2\sin^{-1}\frac{\sqrt{a^2-x^2}}{a}\right\}dx$$

$$=\frac{1}{2}\left[-\frac{1}{3}\left(\sqrt{a^2-x^2}\right)^3+a^2x\sin^{-1}\frac{\sqrt{a^2-x^2}}{a}\right]_0^a-\frac{a^2}{2}\int_0^a x\frac{1}{\sqrt{1-\frac{a^2-x^2}{a^2}}}\frac{1}{a}\frac{-x}{\sqrt{a^2-x^2}}dx$$

$$=\frac{1}{6}a^3+\frac{a^2}{2}\int_0^a\frac{x}{\sqrt{a^2-x^2}}dx=\frac{1}{6}a^3+\frac{a^2}{2}\left[-\sqrt{a^2-x^2}\right]_0^a=\frac{1}{6}a^3+\frac{1}{2}a^3=\frac{2}{3}a^3.$$

(4) $z=x^2+y^2$, $z=y$

$D:\ x^2+y^2-y\leqq 0$
$\quad x^2+(y-\frac{1}{2})^2\leqq\frac{1}{4}$

$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$ $\quad J=r$ $\quad r\leqq\sin\theta$

$$V=\iint_D\{y-(x^2+y^2)\}dx\,dy$$

$$=\int_0^{\pi}d\theta\int_0^{\sin\theta}(r\sin\theta-r^2)r\,dr$$

$$=\int_0^{\pi}\left[\frac{r^3}{3}\sin\theta-\frac{r^4}{4}\right]_0^{\sin\theta}d\theta=\frac{1}{12}\int_0^{\pi}\sin^4\theta\,d\theta=\frac{1}{12}\cdot 2\cdot\frac{3}{4}\cdot\frac{\pi}{4}=\frac{\pi}{32}$$

P.65

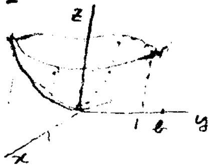
(5) $z=x^2+y^2$, $z=2x+2y+1$ $\qquad x^2+y^2-2x-2y-1\leqq 0$

$V=\iint_D (2x+2y+1-x^2-y^2)\,dx\,dy$ $\qquad D;\ (x-1)^2+(y-1)^2\leqq 3$

$=\iint_D \{3-(x-1)^2-(y-1)^2\}\,dx\,dy$ $\qquad \begin{cases} x-1=r\cos\theta \\ y-1=r\sin\theta \end{cases}$ $\quad J=r$

$=\int_0^{2\pi}d\theta\int_0^{\sqrt3}\{3-r^2\}r\,dr = 2\pi\left[\frac32 r^2-\frac14 r^4\right]_0^{\sqrt3}=\frac92\pi$

(6) $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$, $x^2+y^2=1$, $z=0$

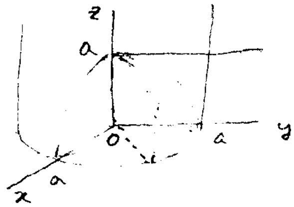


$V=\iint_D\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)dx\,dy$ $\qquad D;\ x^2+y^2\leqq 1$

$=\int_0^{2\pi}d\theta\int_0^1\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)r^3\,dr$ $\qquad x=r\cos\theta$, $y=r\sin\theta$ $\quad J=r$

$=\frac14\int_0^{2\pi}\left\{\frac{1}{a^2}\cos^2\theta+\frac{1}{b^2}\sin^2\theta\right\}d\theta=\frac14\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\pi$

(7) $x^2+y^2=a^2$, $y^2+z^2=a^2$, $z^2+x^2=a^2$



$V=16\iint_D\sqrt{a^2-x^2}\,dx\,dy$ $\qquad D:\ x^2+y^2\leqq a^2$, $0\leqq y\leqq x$

$=16\int_0^{\frac{\pi}{4}}d\theta\int_0^a\sqrt{a^2-r^2\cos^2\theta}\,r\,dr$ $\qquad x=r\cos\theta$, $y=r\sin\theta$

$=16\int_0^{\frac{\pi}{4}}\left[-\frac{1}{3\cos^2\theta}(a^2-r^2\cos^2\theta)^{\frac32}\right]_0^a d\theta$ $\qquad J=r \qquad 0\leqq r\leqq a,\ 0\leqq\theta\leqq\frac{\pi}{4}$

$=\frac{16}{3}\int_0^{\frac{\pi}{4}}\left(\frac{a^3}{\cos^2\theta}-\frac{a^3\sin^3\theta}{\cos^2\theta}\right)d\theta = \frac{16}{3}a^3\int_0^{\frac{\pi}{4}}\left(\sec^2\theta+\sin\theta-\frac{\sin\theta}{\cos^2\theta}\right)d\theta$

$=\frac{16}{3}a^3\left[\tan\theta-\cos\theta-\frac{1}{\cos\theta}\right]_0^{\frac{\pi}{4}}$

$=\frac{16}{3}a^3\left(1-\frac{1}{\sqrt2}+1-\sqrt2+1\right)=\frac{16}{3}a^3\left(3-\frac{3\sqrt2}{2}\right)=8a^3(2-\sqrt2)$

(8) $x^2+y^2+z^2=a^2$, $x^2+y^2=\frac{a^2}{4}$

$V=\iint_D 2\sqrt{a^2-x^2-y^2}\,dx\,dy$ $\qquad D:\ x^2+y^2\leqq\frac{a^2}{4}$

$=\int_0^{2\pi}d\theta\int_0^{\frac{a}{2}}2\sqrt{a^2-r^2}\,r\,dr$ $\qquad x=r\cos\theta$, $y=r\sin\theta$ $\quad J=r$ $\quad 0\leqq r\leqq\frac{a}{2}$

$=2\pi\left[2\left(-\frac13\right)(\sqrt{a^2-r^2})^3\right]_0^{\frac{a}{2}}$

$=\frac43\pi\left(a^3-\frac{3\sqrt3}{8}a^3\right)=\pi a^3\left(\frac43-\frac{\sqrt3}{2}\right)$

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5.2 (1) $0 \le z \le 1-(x^2+y^2)^{\frac{1}{4}}$ D: $x^2+y^2 \le 1$

$V = \iint_D \{1-(x^2+y^2)^{\frac{1}{4}}\}\,dx\,dy$ $x = r\cos\theta$ $y = r\sin\theta$ $0 \le r \le 1$

$J = r$

$= \int_0^{2\pi} d\theta \int_0^1 (1-r^{\frac{1}{2}})\,r\,dr$

$= 2\pi \left[\frac{r^2}{2} - \frac{2}{5}r^{\frac{5}{2}}\right]_0^1 = \frac{\pi}{5}$

(2) $x^2+y^2+z^2 \le 2$, $x^2+y^2 \le 1$ D: $x^2+y^2 \le 1$

$V = \iint_D 2\sqrt{2-x^2-y^2}\,dx\,dy$ $x = r\cos\theta$, $y = r\sin\theta$ $0 \le r \le 1$

$J = r$

$= 2\int_0^{2\pi} d\theta \int_0^1 \sqrt{2-r^2}\,r\,dr$

$= 2\int_0^{2\pi} \left[-\frac{1}{3}(2-r^2)^{\frac{3}{2}}\right]_0^1 d\theta = 2\int_0^{2\pi}\left(\frac{2\sqrt{2}}{3} - \frac{1}{3}\right)d\theta = \frac{4\pi}{3}(2\sqrt{2}-1)$

(3) $x^2+y^2 \le 1$, $z \ge -y$, $z \le 2y$ $\therefore$ $-y \le z \le 2y$

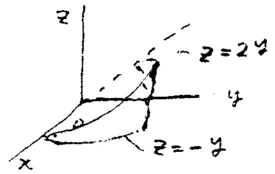
$V = \iint_D 3y\,dx\,dy$ D: $x^2+y^2 \le 1$ $x = r\cos\theta$ $0 \le r \le 1$

$y = r\sin\theta$ $J = r$

$= 2\int_0^{\pi} d\theta \int_0^1 3r^2 \sin\theta\,dr$

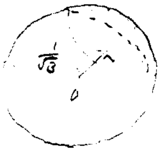
$= 6\int_0^{\pi} \sin\theta\,d\theta \int_0^1 r^2\,dr$

$= 6\left[-\cos\theta\right]_0^{\pi}\left[\frac{r^3}{3}\right]_0^1 = 4$



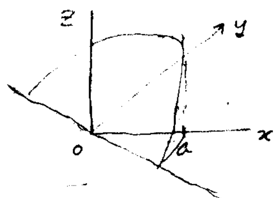
(4) $x^2+y^2+z^2 \le 1$ $x+y+z = 1$

$V = \int_{\frac{1}{\sqrt{3}}}^1 \pi(1-x^2)\,dx$ $\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = \frac{1}{\sqrt{3}}$



$= \pi\left[x - \frac{x^3}{3}\right]_{\frac{1}{\sqrt{3}}}^1 = \pi\left(\frac{2}{3} - \frac{1}{\sqrt{3}} + \frac{1}{9\sqrt{3}}\right) = \pi\left(\frac{2}{3} - \frac{8}{9\sqrt{3}}\right) = \frac{2\pi}{9\sqrt{3}}(3\sqrt{3}-4)$

(5) $x^2+y^2 = a^2$, $0 \le z \le x+y$



$\iint_D (x+y)\,dx\,dy$ D: $x^2+y^2 \le a^2$, $0 \le x+y$

$= \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} d\theta \int_0^a r^2\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)dr$ $x = r\cos\theta$, $y = r\sin\theta$

$0 \le r \le a$, $0 \le r(\cos\theta+\sin\theta) = \sqrt{2}\,r\sin\left(\theta+\frac{\pi}{4}\right)$

$= \sqrt{2}\left[-\cos\left(\theta+\frac{\pi}{4}\right)\right]_{-\frac{\pi}{4}}^{\frac{3}{4}\pi}\left[\frac{r^3}{3}\right]_0^a$ $0 \le \sin\left(\theta+\frac{\pi}{4}\right)$ $-\frac{\pi}{4} \le \theta \le \frac{3}{4}\pi$

$= 2\sqrt{2}\,\frac{a^3}{3} = \frac{2\sqrt{2}}{3}a^3$

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(6) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$ $\quad z=\pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

$V=\iint_D 2c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dxdy$ $\quad D;\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$

$x=ar\cos\theta$, $y=br\sin\theta$ $\quad J=abr$

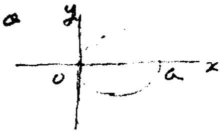
$=\int_0^{2\pi}d\theta\int_0^1 2abc\sqrt{1-r^2}\,r\,dr$

$=2abc\cdot 2\pi\cdot\left[-\frac{1}{3}(1-r^2)^{\frac{3}{2}}\right]_0^1=\frac{4}{3}abc\pi$

(7) $x^2+y^2\le ax$ $\quad x^2+y^2+z^2=a^2$

$V=\iint_D 2\sqrt{a^2-x^2-y^2}\,dxdy$ $\quad z=\pm\sqrt{a^2-x^2-y^2}$ $\quad D;\ x^2+y^2\le ax$

$x=r\cos\theta$, $y=r\sin\theta$ $\quad J=r$ $\quad r\le a\cos\theta$



$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{a\cos\theta}2\sqrt{a^2-r^2}\,r\,dr$

$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2\left[-\frac{1}{3}(a^2-r^2)^{\frac{3}{2}}\right]_0^{a\cos\theta}d\theta$

$=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{3}(a^3-a^3\cos^3\theta)\,d\theta=\frac{2a^3}{3}\cdot 2\left(\frac{\pi}{2}-\frac{2}{3}\right)=\frac{2}{3}a^3\left(\pi-\frac{4}{3}\right)$

(8) $x^2+y^2\le 1,\quad 0\le z\le y$

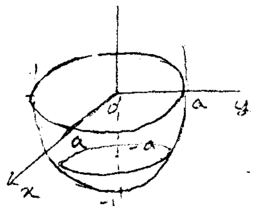
$\iint_D y\,dxdy$ $\quad D:\ x^2+y^2\le 1,\ y\ge 0$

$x=r\cos\theta,\ y=r\sin\theta.\quad J=r\quad 0\le\theta\le\pi$

$=\int_0^{\pi}d\theta\int_0^1 r^2\sin\theta\,dr$

$=\left[-\cos\theta\right]_0^{\pi}\left[\frac{r^3}{3}\right]_0^1=\frac{2}{3}$

5.3 $\quad \frac{x^2}{a^2}+\frac{y^2}{a^2}+z^2\le 1$ $\quad (0<a\le 1)$ $\quad z=-a$



$z=-\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{a^2}}$

$V(a)=\iint_D\left(\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{a^2}}-a\right)dxdy$ $\quad D;\ x^2+y^2\le a^2(1-a^2)$

$x=ar\cos\theta$, $y=ar\sin\theta$ $\quad J=a^2r$

$r^2\le(1-a^2)$ $\quad r\le\sqrt{1-a^2}$

$=\int_0^{2\pi}d\theta\int_0^{\sqrt{1-a^2}}(\sqrt{1-r^2}-a)\,a^2r\,dr.$

$=2\pi a^2\left[-\frac{1}{3}(1-r^2)^{\frac{3}{2}}-a\frac{r^2}{2}\right]_0^{\sqrt{1-a^2}}$

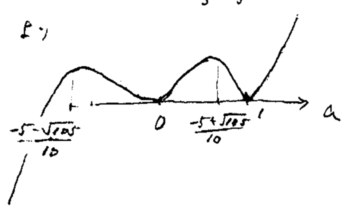
$=2\pi a^2\left(\frac{1}{3}-\frac{1}{3}a^3-\frac{a}{2}+\frac{a^3}{2}\right)=\frac{\pi}{3}a^2(a^3-3a+2)$

$V'(a)=\frac{\pi}{3}(5a^4-9a^2+4a)=\frac{\pi}{3}a(5a^3-9a+4)$

$\begin{array}{rrrr|r} 5 & 0 & -9 & 4 & 1 \\ & 5 & 5 & -4 & \\ \hline 5 & 5 & -4 & 0 & \end{array}$

$=\frac{\pi}{3}a(a-1)(5a^2+5a-4)$ $\quad 0<a\le 1$ より

$a=\frac{-5\pm\sqrt{105}}{10}$, $\quad a=\frac{-5+\sqrt{105}}{10}$



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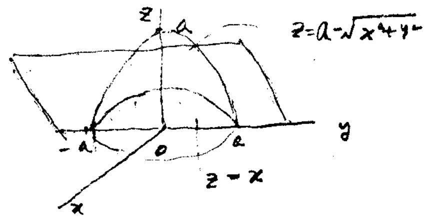
5.4 $z = a - \sqrt{x^2+y^2}$ $x=0$, $z=x$

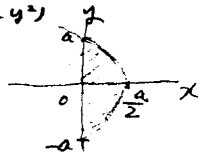
$x = a - \sqrt{x^2+y^2}$ $x^2+y^2 = (a-x)^2$

$y^2 = a^2-2ax$

$x = \frac{a}{2} - \frac{y^2}{2a}$



$V = \iint_D \{a - \sqrt{x^2+y^2} - x\}\,dx\,dy$ D: $0 \leqq x \leqq \frac{1}{2a}(a^2-y^2)$



$= \int_{-a}^{a} dy \int_0^{\frac{1}{2a}(a^2-y^2)} \{(a-x) - \sqrt{x^2+y^2}\}\,dx$

$= \int_{-a}^{a} \left[-\frac{1}{2}(a-x)^2 - \frac{1}{2}\{x\sqrt{x^2+y^2} + y^2 \log(x+\sqrt{x^2+y^2})\}\right]_0^{\frac{1}{2a}(a^2-y^2)} dy$

$= \int_{-a}^{a} \{\frac{1}{2}a^2 - \frac{1}{2}(\frac{a^2-y^2}{2a})^2 - \frac{1}{2}\frac{a^2-y^2}{2a}\cdot\frac{a^2+y^2}{2a} - \frac{y^2}{2}\log(\frac{a^2-y^2}{2a} + \frac{a^2+y^2}{2a}) + \frac{y^2}{2}\log|y|\}dy$

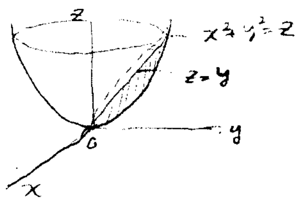
$= \int_{-a}^{a} (\frac{1}{2}a^2 - \frac{1}{2}\frac{a^4+a^2y^2}{2a^2} - \frac{y^2}{2}\log a + \frac{y^2}{2}\log|y|)\,dy$

$= 2\left[\frac{1}{2}a^2 y - \frac{a^2}{4}y - \frac{1}{4}\frac{y^3}{3} - \frac{y^3}{6}\log a + \frac{y^3}{6}\log|y| - \frac{y^3}{18}\right]_0^a$

$= 2(\frac{1}{2}a^3 - \frac{a^3}{4} - \frac{a^3}{12} - \frac{a^3}{18}) = \frac{2}{9}a^3$

5.5 $x^2+y^2 = z$, $z = y$

$x^2+y^2 = y$



$\iint_D \{y - (x^2+y^2)\}\,dx\,dy$ D: $x^2+y^2 \leqq y$

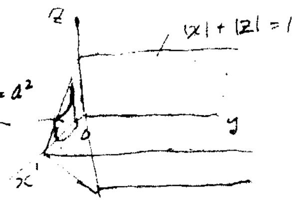
$= \int_0^{\pi} d\theta \int_0^{\sin\theta} (r^2\sin\theta - r^3)\,dr$ $\left[\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right.$ $r \leqq \sin\theta$, $J = r$

$= \int_0^{\pi} (\frac{1}{3}\sin^4\theta - \frac{1}{4}\sin^4\theta)\,d\theta = \frac{1}{12}\int_0^{\pi}\sin^4\theta\,d\theta = \frac{1}{12}\cdot 2\cdot\frac{3}{4}\cdot\frac{\pi}{4} = \frac{\pi}{32}$

5.6 $(x-2a)^2+y^2 \leqq a^2$ $|x|+|z| \leqq 1$

$0 < 3a \leqq 1$

$x-1 \leqq z \leqq 1-x$



$\iint_D \{(1-x) - (x-1)\}\,dx\,dy$ D; $(x-2a)^2+y^2 \leqq a^2$

$y = \pm\sqrt{a^2-(x-2a)^2}$

$= \int_a^{3a} dx \int_{-\sqrt{a^2-(x-2a)^2}}^{\sqrt{a^2-(x-2a)^2}} 2(1-x)\,dy = 4\int_a^{3a}(1-x)\sqrt{a^2-(x-2a)^2}\,dx$

$x - 2a = a\sin\theta$

$dx = a\cos\theta\,d\theta$

$= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-2a-a\sin\theta)\,a^2\cos^2\theta\,d\theta = 4a^2(1-2a)\frac{\pi}{2} + 4a^3\left[\frac{1}{3}\cos^3\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= 2a^2\pi(1-2a)$

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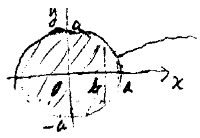
5.7



$y^2+x^2=a^2$ $y^2=a^2-x^2$

$V(b)=\pi\int_{-a}^{b}(a^2-x^2)dx=\pi\left[a^2x-\frac{1}{3}x^3\right]_{-a}^{b}$

$=\pi\left(\frac{2}{3}a^3+a^2b-\frac{1}{3}b^3\right)$

$S(b)=2\pi\int_{-a}^{b}y\sqrt{1+y'^2}\,dx=2\pi\int_{-a}^{b}\sqrt{a^2-x^2}\sqrt{1+\frac{x^2}{a^2-x^2}}\,dx=2\pi a\int_{-a}^{b}dx$

$=2\pi a(b+a)$

∴ $\frac{V(b)}{S(b)}=\frac{1}{2a(b+a)}\left(\frac{2}{3}a^3+a^2b-\frac{1}{3}b^3\right)=\frac{1}{2}\frac{1}{a^2+ab}\left(\frac{2}{3}a^3+a^2b-\frac{1}{3}b^3\right)$

$F(b)=V(b)/S(b)$ とおくと

$F'(b)=\frac{1}{2}\left\{\frac{1}{a^2+ab}(a^2-b^2)-\frac{a}{(a^2+ab)^2}\left(\frac{2}{3}a^3+a^2b-\frac{1}{3}b^3\right)\right\}$

$=\frac{1}{2(a^2+ab)^2}\left\{a^4+a^3b-a^2b^2-ab^3-\frac{2}{3}a^4-a^3b+\frac{1}{3}ab^3\right\}$

$=\frac{1}{2(a^2+ab)^2}\left(\frac{1}{3}a^4-a^2b^2-\frac{2}{3}ab^3\right)=\frac{1}{6a(a+b)^2}(a^3-3ab^2-2b^3)$

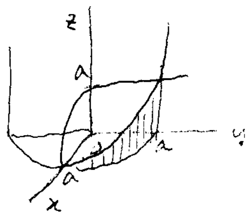
$=\frac{1}{6a(a+b)^2}(a+b)^2(a-2b)$

$=\frac{1}{6a}(a-2b)$ $F'(b)=0$ より $b=\frac{1}{2}a$

$$\begin{array}{cccc|c} 1 & 0 & -3b^2 & -2b^3 & -b \\ & -b & b^2 & 2b^3 & \\ 1 & -b & -2b^2 & 0 & \\ & -b & 2b^2 & & \\ 1 & -2b & 0 & & \end{array}$$

∴ $h=a+b=\frac{3}{2}a$

5.8



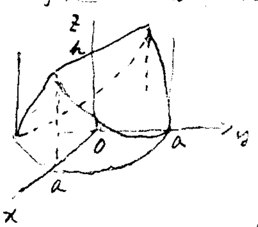
$x^2+z^2=a^2$, $x^2+y^2=a^2$ D: $x^2+y^2\leqq a^2$ $x,y\geqq0$

$V=8\iint_D\sqrt{a^2-x^2}\,dx\,dy$

$=8\int_0^a dx\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2}\,dy=8\int_0^a(a^2-x^2)dx$

$=8\left[a^2x-\frac{x^3}{3}\right]_0^a=\frac{16}{3}a^3$

5.9 問題に図を記入



$x^2+y^2=a^2$, $z=-\frac{h}{a}y+h$

$V=2\iint_D\left(-\frac{h}{a}y+h\right)dx\,dy$ D: $x^2+y^2\leqq a^2$, $y\geqq0$

$x=r\cos\theta$ $J=r$

$y=r\sin\theta$ $0\leqq r\leqq a$

$0\leqq\theta\leqq\pi$

$=2\int_0^\pi d\theta\int_0^a\left(-\frac{h}{a}r^2\sin\theta+hr\right)dr$

$=2\int_0^\pi\left(-\frac{a^2h}{3}\sin\theta+\frac{h}{2}a^2\right)d\theta=2a^2h\left[\frac{1}{3}\cos\theta+\frac{\theta}{2}\right]_0^\pi$

$=2a^2h\left(-\frac{2}{3}+\frac{\pi}{2}\right)$