大日本図書　新微分積分 I 問題集　4 章§ 2 BASIC 詳しい解答

p.52. 4 章§ 2. いろいろな応用　BASIC

226. $S=\displaystyle\int_\alpha^\beta\left|y\frac{dx}{dt}\right|dt$

(1) $(x'=)\dfrac{dx}{dt}=2t, S=\displaystyle\int_0^1\left|y\frac{dx}{dt}\right|dt=\int_0^1|(t^2-2t+1)2t|dt=\int_0^1(2t^3-4t^2+2t)dt=\left[\frac{t^4}{2}-\frac{4t^3}{3}+t^2\right]_0^1$

$=\dfrac{1}{2}-\dfrac{4}{3}+1=\dfrac{1}{6}.$

(別解)$0\leqq t\leqq 1$ のとき $0\leqq x\leqq 1, y\geqq 0$ より $S=\displaystyle\int_0^1 ydx$. $y=t^2-2t+1$. $x=t^2$ より $dx=2tdt$.

$\begin{array}{c|ccc} x & 0 & \to & 1 \\ \hline t & 0 & \to & 1 \end{array}$. $S=\displaystyle\int_0^1(t^2-2t+1)2tdt=\int_0^1(2t^3-4t^2+2t)dt=\left[\frac{t^4}{2}-\frac{4t^3}{3}+t^2\right]_0^1=\frac{1}{2}-\frac{4}{3}+1=\frac{1}{6}.$

(2) $(x'=)\dfrac{dx}{dt}=\dfrac{1}{\cos^2 t}, S=\displaystyle\int_0^{\frac{\pi}{4}}\left|y\frac{dx}{dt}\right|dt=\int_0^{\frac{\pi}{4}}\left|\frac{\sin t+1}{\cos^2 t}\right|dt=\int_0^{\frac{\pi}{4}}\frac{\sin t}{\cos^2 t}dt+\int_0^{\frac{\pi}{4}}\frac{dt}{\cos^2 t}.$

第 1 項で $\cos t=u$ とおくと $-\sin t=du\Rightarrow \sin tdt=-du$. $\begin{array}{c|ccc} t & 0 & \to & \frac{\pi}{4} \\ \hline u & 1 & \to & \frac{1}{\sqrt{2}} \end{array}$.

$S=\displaystyle\int_1^{\frac{1}{\sqrt{2}}}\frac{1}{u^2}(-du)+\int_0^{\frac{\pi}{4}}\frac{dt}{\cos^2 t}=-\left[\frac{u^{-1}}{-1}\right]_1^{\frac{1}{\sqrt{2}}}+[\tan t]_0^{\frac{\pi}{4}}=\frac{1}{\frac{1}{\sqrt{2}}}-1+\tan\frac{\pi}{4}=\sqrt{2}.$

(別解)$0\leqq t\leqq \dfrac{\pi}{4}$ のとき $0\leqq x\leqq 1, y\geqq 0$ より $S=\displaystyle\int_0^1 ydx$. $y=\sin t+1$. $x=\tan t$ より $dx=\dfrac{dt}{\cos^2 t}$.

$\begin{array}{c|ccc} x & 0 & \to & 1 \\ \hline t & 0 & \to & \frac{\pi}{4} \end{array}$. $S=\displaystyle\int_0^{\frac{\pi}{4}}\frac{\sin t+1}{\cos^2 t}dt$. 以下は同様.

(3) $(x'=)\dfrac{dx}{dt}=-\sin t, S=\displaystyle\int_0^{\frac{\pi}{2}}\left|y\frac{dx}{dt}\right|dt=\int_0^{\frac{\pi}{2}}|(\cos 2t+1)(-\sin t)|\,dt=\int_0^{\frac{\pi}{2}}(\cos 2t+1)\sin tdt.$

$\cos 2t=2\cos^2-1$ (2 倍角の公式) より $\cos t=u$ とおくと $-\sin tdt=du\Rightarrow \sin tdt=-du$. $\begin{array}{c|ccc} t & 0 & \to & \frac{\pi}{2} \\ \hline u & 1 & \to & 0 \end{array}$.

$S=\displaystyle\int_0^{\frac{\pi}{2}}2\cos^2 t\sin tdt=\int_1^0 2u^2(-du)=-\left[\frac{2u^3}{3}\right]_1^0=0-\left(-\frac{2}{3}\right)=\frac{2}{3}.$

(別解)$0\leqq t\leqq \dfrac{\pi}{2}$ のとき $0\leqq x\leqq 1, y\geqq 0$ より $S=\displaystyle\int_0^1 ydx$.

2 倍角の公式より $y=\cos 2t+1=2\cos^2 t-1+1=2\cos^2 t=2x^2$. $S=\displaystyle\int_0^1 2x^2dx=\left[\frac{2x^3}{3}\right]_0^1=\frac{2}{3}.$

227. $l=\displaystyle\int_\alpha^\beta\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt$

(1) $(x'=)\dfrac{dx}{dt}=6t, (y'=)\dfrac{dy}{dt}=3-3t^2$. よって $\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2=(6t)^2+(3-3t^2)^2=36t^2+9-18t^2+9t^4$

$=9t^4+18t^2+9=9(t^2+1)^2$. よって $l=\displaystyle\int_0^{\sqrt{3}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt=\int_0^{\sqrt{3}}3(t^2+1)dt=[t^3+3t]_0^{\sqrt{3}}$

$=3\sqrt{3}+3\sqrt{3}=6\sqrt{3}.$

(2) $(x'=)\dfrac{dx}{dt}=-\sin t+\sin t+t\cos t=t\cos t, (y'=)\dfrac{dy}{dt}=\cos t-\cos t-t(-\sin t)=t\sin t$. よって

$\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2=(t\cos t)^2+(t\sin t)^2=t^2(\cos^2 t+\sin^2 t)=t^2.$

よって $l=\displaystyle\int_0^\pi\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt=\int_0^\pi tdt=\left[\frac{t^2}{2}\right]_0^\pi=\frac{\pi^2}{2}$

(3) $(x'=)\dfrac{dx}{dt}=-2\sin t+2\sin 2t, (y'=)\dfrac{dy}{dt}=2\cos t-2\cos 2t$. よって $\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2=(-2\sin t+2\sin 2t)^2+$

$(2\cos t-2\cos 2t)^2=4\sin^2 t-8\sin t\sin 2t+4\sin^2 2t+4\cos^2 t-8\cos t\cos 2t+4\cos^2 2t$

$=4(\sin^2 t+\cos^2 t)-8(\sin t\sin 2t+\cos t\cos 2t)+4(\sin^2 2t+\cos^2 2t)=4-8\cos(2t-t)+4=8(1-\cos t).$

(加法定理を逆に使って $\cos\alpha\cos\beta+\sin\alpha\sin\beta=\cos(\alpha-\beta)$ $(\alpha=2t,\beta=t)$ とした)

半角の公式 $\sin^2\dfrac{\alpha}{2}=\dfrac{1-\cos\alpha}{2}$ より $1-\cos\alpha=2\sin^2\dfrac{\alpha}{2}$ だから $8(1-\cos t)=16\sin^2\dfrac{t}{2}$. よって

$l=\displaystyle\int_0^{\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_0^{\pi}4\sin\frac{t}{2}dt=\left[-8\cos\frac{t}{2}\right]_0^{\pi}=-8\cos\frac{\pi}{2}-(-8\cos 0)=8.$

228. (1) $(x'=)\dfrac{dx}{dt}=-a\sin t, S=\displaystyle\int_0^{2\pi}\left|y\frac{dx}{dt}\right|dt=\int_0^{2\pi}|b\sin t(-a\sin t)|dt=ab\int_0^{2\pi}\sin^2 t\,dt=ab\int_0^{2\pi}\frac{1-\cos 2t}{2}dt$

$=\dfrac{ab}{2}\left[t-\dfrac{\sin 2t}{2}\right]_0^{2\pi}=\dfrac{ab}{2}\cdot 2\pi=\pi ab.$ (半角の公式より $\sin^2 t=\dfrac{1-\cos 2t}{2}$ を用いた)

(2) 回転体の体積の公式 $V=\pi\displaystyle\int_\alpha^\beta y^2\left|\frac{dx}{dt}\right|dt$

$(x'=)\dfrac{dx}{dt}=-a\sin t, V=\pi\displaystyle\int_0^{\pi}y^2\left|\frac{dx}{dt}\right|dt=\pi\int_0^{\pi}b^2\sin^2 t|-a\sin t|dt=\pi ab^2\int_0^{\pi}\sin^3 t\,dt$

$=\pi ab^2\displaystyle\int_0^{\pi}(1-\cos^2 t)\sin t\,dt.$ $\cos t=u$ とおくと $-\sin t\,dt=du\Rightarrow\sin t\,dt=-du.$ $\begin{array}{c|ccc} t & 0 & \to & \pi \\ \hline u & 1 & \to & -1 \end{array}.$

$V=\pi ab^2\displaystyle\int_1^{-1}(1-u^2)(-du)=2\pi ab^2\int_0^1(1-u^2)du=2\pi ab^2\left[u-\frac{u^3}{3}\right]_0^1=2\pi ab^2\left(1-\frac{1}{3}\right)=\frac{4}{3}\pi ab^2.$

229. (1) $(x'=)\dfrac{dx}{dt}=\dfrac{1}{2}t^{-\frac{1}{2}}=\dfrac{1}{2\sqrt{t}}, V=\pi\displaystyle\int_0^1 y^2\left|\frac{dx}{dt}\right|dt=\pi\int_0^1(\sqrt{t}-t)^2\left|\frac{1}{2\sqrt{t}}\right|dt=\pi\int_0^1\frac{t-2t\sqrt{t}+t^2}{2\sqrt{t}}dt$

$=\dfrac{\pi}{2}\displaystyle\int_0^1(t^{\frac{1}{2}}-2t+t^{\frac{3}{2}})dt=\frac{\pi}{2}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}-t^2+\frac{t^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1=\frac{\pi}{2}\left(\frac{2}{3}-1+\frac{2}{5}\right)=\frac{\pi}{30}.$

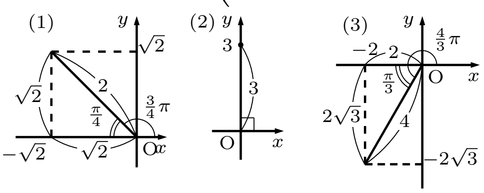
(2) $(x'=)\dfrac{dx}{dt}=\cos t, V=\pi\displaystyle\int_0^{\frac{\pi}{2}}y^2\left|\frac{dx}{dt}\right|dt=\pi\int_0^{\frac{\pi}{2}}\sin^2 2t|\cos t|dt=4\pi\int_0^{\frac{\pi}{2}}\sin^2 t\cos^3 t\,dt$

$=4\pi\displaystyle\int_0^{\frac{\pi}{2}}(1-\cos^2 t)\cos^3 t\,dt=4\pi\left(\int_0^{\frac{\pi}{2}}\cos^3 t\,dt-\int_0^{\frac{\pi}{2}}\cos^5 t\,dt\right)=4\pi\left(\frac{2}{3}-\frac{4}{5}\cdot\frac{2}{3}\right)=\frac{8}{15}\pi.$

230. (1) $(-\sqrt{2},\sqrt{2})$

(2) $(0,3)$

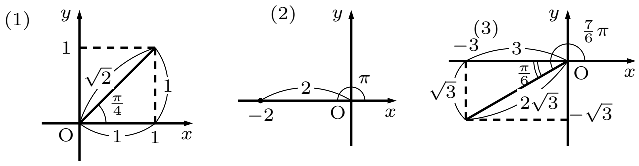
(3) $(-2,-2\sqrt{3})$



231. (1) $\left(\sqrt{2},\dfrac{\pi}{4}\right)$

(2) $(2,\pi)$

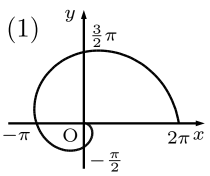
(3) $\left(2\sqrt{3},\dfrac{7}{6}\pi\right)$



232. (1) 原点中心の半径 1 の円

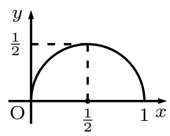
(2) x 軸の正の向きと $-\dfrac{\pi}{6}$ $\left(\text{下向き}\dfrac{\pi}{6}\right)$ の角を作る原点から始まる半直線 (原点は含まない).

233. (1)



(2) $r^2=r\cos\theta\Rightarrow x^2+y^2=x\Rightarrow x^2+y^2-x=0\Rightarrow\left(x-\dfrac{1}{2}\right)^2+y^2=\left(\dfrac{1}{2}\right)^2$

$\Rightarrow$ 中心 $\left(\dfrac{1}{2},0\right)$ 半径 $\dfrac{1}{2}$ の円のうち $0\leqq\theta\leqq\dfrac{\pi}{2}$ より第 1 象限の部分.



234. $S=\dfrac{1}{2}\displaystyle\int_\alpha^\beta r^2d\theta$

(1) $S=\dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{2}}r^2d\theta=\frac{1}{2}\int_0^{\frac{\pi}{2}}\theta^4d\theta=\frac{1}{2}\left[\frac{\theta^5}{5}\right]_0^{\frac{\pi}{2}}=\frac{1}{10}\left(\frac{\pi}{2}\right)^5=\frac{\pi^5}{320}.$

(2) $S=\dfrac{1}{2}\displaystyle\int_0^{2\pi}r^2d\theta=\frac{1}{2}\int_0^{2\pi}(\cos\theta+2)^2d\theta=\frac{1}{2}\int_0^{2\pi}(\cos^2\theta+4\cos\theta+4)d\theta=\frac{1}{2}\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}+4\cos\theta+4\right)d\theta$

$=\dfrac{1}{2}\left[\dfrac{\theta}{2}+\dfrac{\sin 4\theta}{4}+4\sin\theta+4\theta\right]_0^{2\pi}=\dfrac{1}{2}(\pi+8\pi)=\dfrac{9}{2}\pi.$

235. $l=\displaystyle\int_\alpha^\beta\sqrt{r^2+(r')^2}d\theta$

(1) $l=\displaystyle\int_0^{\pi}\sqrt{(e^\theta)^2+\{(e^\theta)'\}^2}d\theta=\int_0^{\pi}\sqrt{2}e^\theta d\theta=\sqrt{2}\left[e^\theta\right]_0^{\pi}=\sqrt{2}\,(e^\pi-1).$

(2) $l=\int_0^{\frac{\pi}{2}}\sqrt{r^2+(r')^2}d\theta=\int_0^{\frac{\pi}{2}}\sqrt{\cos^2\theta+\{(\cos\theta)'\}^2}d\theta=\int_0^{\frac{\pi}{2}}\sqrt{\cos^2\theta+(-\sin\theta)^2}d\theta=\int_0^{\frac{\pi}{2}}\sqrt{\cos^2\theta+\sin^2\theta}d\theta$
$=\int_0^{\frac{\pi}{2}}d\theta=[\theta]_0^{\frac{\pi}{2}}=\frac{\pi}{2}.$

236. (1) 与式 $=\lim_{\varepsilon\to+0}\int_{2+\varepsilon}^3\frac{dx}{\sqrt{x-2}}=\lim_{\varepsilon\to+0}\left[\frac{(x-2)^{\frac{1}{2}}}{\frac{1}{2}}\right]_{2+\varepsilon}^3=\lim_{\varepsilon\to+0}\left[2\sqrt{x-2}\right]_{2+\varepsilon}^3=\lim_{\varepsilon\to+0}\left(2\sqrt{1}-2\sqrt{\varepsilon}\right)=2.$
(与式 $=\left[2\sqrt{x-2}\right]_2^3=2.$ でもよい)
(2) 与式 $=\left[\sin^{-1}\frac{x}{3}\right]_0^3=\sin^{-1}1=\frac{\pi}{2}.$ (公式 $\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}$ を用いた)
(3) 与式 $=\lim_{\varepsilon\to+0}\int_\varepsilon^1\frac{dx}{\sqrt[4]{x}}=\lim_{\varepsilon\to+0}\left[\frac{x^{\frac{3}{4}}}{\frac{3}{4}}\right]_\varepsilon^1=\frac{4}{3}\lim_{\varepsilon\to+0}\left[\sqrt[4]{x^3}\right]_\varepsilon^1=\frac{4}{3}\lim_{\varepsilon\to+0}\left(\sqrt[4]{1}-\sqrt[4]{\varepsilon^3}\right)=\frac{4}{3}.$
(与式 $=\frac{4}{3}\left[\sqrt[4]{x^3}\right]_0^1=\frac{4}{3}.$ でもよい)
(4) 与式 $=2\lim_{\varepsilon\to+0}\int_\varepsilon^1\frac{dx}{x^4}=2\lim_{\varepsilon\to+0}\left[\frac{x^{-3}}{-3}\right]_\varepsilon^1=2\lim_{\varepsilon\to+0}\left[-\frac{1}{3x^3}\right]_\varepsilon^1=2\lim_{\varepsilon\to+0}\left(-\frac{1}{3}+\frac{1}{3\varepsilon^3}\right)=\infty.$ よって存在しない.

237. $S=\int_{-a}^a\frac{b}{a}\sqrt{a^2-x^2}dx=\frac{2b}{a}\int_0^a\sqrt{a^2-x^2}dx=\frac{2b}{a}\cdot\frac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}\right]_0^a=\frac{b}{a}\cdot a^2\sin^{-1}1=\frac{\pi ab}{2}.$

238. (1) 与式 $=\lim_{b\to\infty}\int_2^b x^{-5}dx=\lim_{b\to\infty}\left[\frac{x^{-4}}{-4}\right]_2^b=\lim_{b\to\infty}\left[-\frac{1}{4x^4}\right]_2^b=\lim_{b\to\infty}\left(-\frac{1}{4b^4}+\frac{1}{64}\right)=\frac{1}{64}.$
(与式 $=\left[-\frac{1}{4x^4}\right]_2^\infty=\frac{1}{64}.$ でもよい)
(2) 与式 $=\lim_{b\to\infty}\int_0^b\frac{dx}{e^{2x}}=\lim_{b\to\infty}\left[\frac{e^{-2x}}{-2}\right]_0^b=\lim_{b\to\infty}\left[-\frac{1}{2e^{2x}}\right]_0^b=\lim_{b\to\infty}\left(-\frac{1}{2e^{2b}}+\frac{1}{2}\right)=\frac{1}{2}.$
(与式 $=\left[-\frac{1}{2e^{2x}}\right]_0^\infty=\frac{1}{2}.$ でもよい)
(3) 与式 $=\lim_{b\to\infty}\int_0^b\frac{dx}{\sqrt[3]{x^2}}=\lim_{b\to\infty}\left[\frac{x^{\frac{1}{3}}}{\frac{1}{3}}\right]_0^b=\lim_{b\to\infty}\left[3\sqrt[3]{x}\right]_0^b=\lim_{b\to\infty}\left(3\sqrt[3]{b}-0\right)=\infty.$ よって存在しない.
(4) 与式 $=\lim_{b\to\infty}\int_0^b xe^{-x}dx=\lim_{b\to\infty}\left(\left[x(-e^{-x})\right]_0^b-\int_0^b 1\cdot(-e^{-x})dx\right)=\lim_{b\to\infty}\left(-be^{-b}+\int_0^b e^{-x}dx\right)$
$=\lim_{b\to\infty}\left(-\frac{b}{e^b}+\left[-e^{-x}\right]_0^b\right)=\lim_{b\to\infty}\left(-\frac{b}{e^b}-e^{-b}+1\right).$ ロピタルの定理より $\lim_{b\to\infty}\frac{b}{e^b}=\lim_{b\to\infty}\frac{(b)'}{(e^b)'}=\lim_{b\to\infty}\frac{1}{e^b}=0.$
また $\lim_{b\to\infty}e^{-b}=\lim_{b\to\infty}\frac{1}{e^b}=0$ だから与式 $=1.$

239. $x(t)=x(a)+\int_a^t v(t)dt,\ v(t)=v(a)+\int_a^t\alpha(t)dt$ ここでは $x(0)=0, v(0)=0.$
(1) $v(t)=v(0)+\int_0^t\alpha(t)dt=0+\int_0^t\left\{-18\sin\left(3t+\frac{\pi}{4}\right)\right\}dt=\left[6\cos\left(3t+\frac{\pi}{4}\right)\right]_0^t=6\cos\left(3t+\frac{\pi}{4}\right)-6\cos\frac{\pi}{4}$
$=6\cos\left(3t+\frac{\pi}{4}\right)-3\sqrt{2}.$
(2) $x(t)=x(0)+\int_0^t v(t)dt=0+\int_0^t\left\{6\cos\left(3t+\frac{\pi}{4}\right)-3\sqrt{2}\right\}dt=\left[2\sin\left(3t+\frac{\pi}{4}\right)-3\sqrt{2}t\right]_0^t$
$=2\sin\left(3t+\frac{\pi}{4}\right)-3\sqrt{2}t-2\sin\frac{\pi}{4}=2\sin\left(3t+\frac{\pi}{4}\right)-3\sqrt{2}t-\sqrt{2}.$

240. 単位時間内に減少する質量, すなわち減少率は $-x'(t)$(減少だから $-$) で $-x'(t)=kx(t)$. よって
$\frac{x'(t)}{x(t)}=-k.$ $\int\frac{x'(t)}{x(t)}dt=-\int kdt.$ $\log|x(t)|=-kt+c.$ $x(t)=e^{-kt+c}=e^{-kt}e^c.$
$t=0$ とすると $x(0)=e^c$. $x(0)=x_0$ より $e^c=x_0$. よって $x(t)=x_0e^{-kt}.$

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241. (1) $\dfrac{dx}{dt}=3t^2$. $S=\displaystyle\int_0^2\left|y\frac{dx}{dt}\right|dt=\int_0^2|(t-2)^2\cdot 3t^2|dt=3\int_0^2(t^4-4t^3+4t^2)dt=3\left[\frac{t^5}{5}-t^4+\frac{4}{3}t^3\right]_0^2$

$=3\left(\dfrac{32}{5}-16+\dfrac{32}{3}\right)=\dfrac{16}{5}$.

(2) $\dfrac{dx}{dt}=e^t$. $S=\displaystyle\int_0^1\left|y\frac{dx}{dt}\right|dt=\int_0^1|(e^{2t}+1)e^t|dt=\int_0^1(e^{3t}+e^t)dt=\left[\frac{1}{3}e^{3t}+e^t\right]_0^1=\frac{e^3}{3}+e-\frac{1}{3}-1$

$=\dfrac{e^3}{3}+e-\dfrac{4}{3}$.

242. (1) $\dfrac{dx}{dt}=3t^2,\dfrac{dy}{dt}=6t$. $l=\displaystyle\int_0^1\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2}dt=\int_0^1\sqrt{9t^4+36t^2}dt=\int_0^1 3t\sqrt{t^2+4}dt.$

$t^2+4=u$ とおくと $2tdt=du, tdt=\dfrac{du}{2}$. $\begin{array}{c|ccc} t & 0 & \to & 1 \\ \hline u & 4 & \to & 5 \end{array}$ $l=3\displaystyle\int_4^5\sqrt{u}\frac{du}{2}=\frac{3}{2}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_4^5=5^{\frac{3}{2}}-4^{\frac{3}{2}}=5\sqrt{5}-8.$

(2) $\dfrac{dx}{dt}=-e^{-t}\cos t+e^t(-\sin t), \dfrac{dy}{dt}=-e^{-t}\sin t+e^{-t}\cos t.$

$\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dx}{dt}\right)^2=\{-e^{-t}(\cos t+\sin t)\}^2+\{e^{-t}(-\sin t+\cos t)\}^2$

$=e^{-2t}(\cos^2 t+2\sin t\cos t+\sin^2 t)+e^{-2t}(\sin^2 t-2\sin t\cos t+\cos^2 t)=2e^{-2t}(\cos^2 t+\sin^2 t)=2e^{-2t}.$

$l=\displaystyle\int_0^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2}dt=\int_0^{2\pi}\sqrt{2e^{-2t}}dt=\sqrt{2}\int_0^{2\pi}e^{-t}dt=\sqrt{2}[-e^{-t}]_0^{2\pi}=\sqrt{2}(-e^{-2\pi}+1)$

$=\sqrt{2}\left(1-\dfrac{1}{e^{2\pi}}\right).$

243. (1) $\dfrac{dx}{dt}=3t^2$. $V=\pi\displaystyle\int_{-1}^1 y^2\left|\frac{dx}{dt}\right|dt=\pi\int_{-1}^1(t^2-1)^2|3t^2|dt=6\pi\int_0^1(t^6-2t^4+t^2)dt=6\pi\left[\frac{t^7}{7}-\frac{2}{5}t^5+\frac{t^3}{3}\right]_0^1$

$=6\pi\left(\dfrac{1}{7}-\dfrac{2}{5}+\dfrac{1}{3}\right)=\dfrac{16}{35}\pi.$

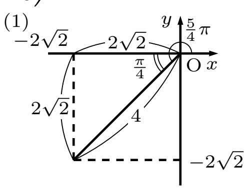
(2) $\dfrac{dx}{dt}=2t$. $V=\pi\displaystyle\int_0^1 y^2\left|\frac{dx}{dt}\right|dt=\pi\int_0^1(e^t)^2|2t|dt=2\pi\int_0^1 te^{2t}dt=2\pi\left(\left[t\cdot\frac{e^{2t}}{2}\right]_0^1-\int_0^1\frac{e^{2t}}{2}dt\right)$

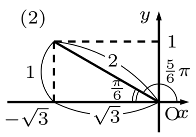
$=2\pi\left(\dfrac{e^2}{2}-\left[\dfrac{e^{2t}}{4}\right]_0^1\right)=\pi\left(e^2-\dfrac{e^2-1}{2}\right)=\dfrac{\pi}{2}(e^2+1).$

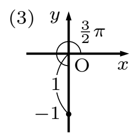
244. (1) $(-2\sqrt{2},-2\sqrt{2})$.

(2) $(-\sqrt{3},1)$.

(3) $(0,-1)$.



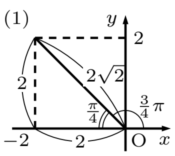


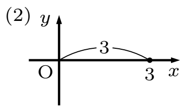


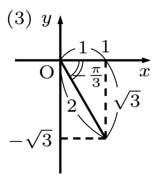
245. (1) $\left(2\sqrt{2},\dfrac{3}{4}\pi\right)$

(2) $(3,0)$

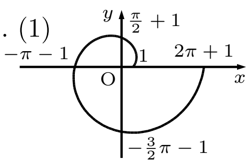
(3) $\left(2,-\dfrac{\pi}{3}\right)$





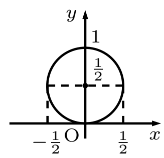


246. (1)



(2) $r^2=r\sin\theta\Rightarrow x^2+y^2=y\Rightarrow x^2+y^2-y=0$

$\Rightarrow x^2+\left(y-\dfrac{1}{2}\right)^2=\left(\dfrac{1}{2}\right)^2\Rightarrow$ 中心 $\left(0,\dfrac{1}{2}\right)$ 半径 $\dfrac{1}{2}$ の円.



247. (1) $S=\dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{2}}r^2d\theta=\frac{1}{2}\int_0^{\frac{\pi}{2}}(1+\theta)^2d\theta=\frac{1}{2}\int_0^{\frac{\pi}{2}}(1+2\theta+\theta^2)d\theta=\frac{1}{2}\left[\theta+\theta^2+\frac{\theta^3}{3}\right]_0^{\frac{\pi}{2}}=\frac{\pi}{4}+\frac{\pi^2}{8}+\frac{\pi^3}{48}.$

(2) $S=\dfrac{1}{2}\displaystyle\int_0^{2\pi}r^2d\theta=\frac{1}{2}\int_0^{2\pi}|\cos\theta|^2d\theta=\frac{1}{2}\int_0^{2\pi}\cos^2\theta d\theta=\frac{1}{2}\int_0^{2\pi}\frac{1+\cos 2\theta}{2}d\theta=\frac{1}{2}\left[\frac{\theta}{2}+\frac{\sin 4\theta}{4}\right]_0^{2\pi}=\frac{\pi}{2}.$

248. (1) $l=\displaystyle\int_0^{\frac{\pi}{2}}\sqrt{(\cos\theta)^2+\{(\cos\theta)'\}^2}d\theta=\int_0^{\frac{\pi}{2}}\sqrt{\cos^2\theta+\sin^2\theta}d\theta=\int_0^{\frac{\pi}{2}}d\theta=[\theta]_0^{\frac{\pi}{2}}=\frac{\pi}{2}.$

(2) $l=\displaystyle\int_0^{4\pi}\sqrt{\left(\sin^4\frac{\theta}{4}\right)^2+\left\{\left(\sin^4\frac{\theta}{4}\right)'\right\}^2}d\theta=\int_0^{4\pi}\sqrt{\sin^8\frac{\theta}{4}+\left\{\left(4\sin^3\frac{\theta}{4}\cos\frac{\theta}{4}\right)\cdot\frac{1}{4}\right\}^2}d\theta$

$=\displaystyle\int_0^{4\pi}\sqrt{\sin^8\frac{\theta}{4}+\sin^6\frac{\theta}{4}\cos^2\frac{\theta}{4}}d\theta=\int_0^{4\pi}\sqrt{\sin^6\frac{\theta}{4}\left(\sin^2\frac{\theta}{4}+\cos^2\frac{\theta}{4}\right)}d\theta=\int_0^{4\pi}\sqrt{\sin^6\frac{\theta}{4}}d\theta.$

$0\leqq\theta\leqq 4\pi$ より $0\leqq\dfrac{\theta}{4}\leqq\pi$. よって $\sin\dfrac{\pi}{4}\geqq 0$. 従って $l=\displaystyle\int_0^{4\pi}\sin^3\frac{\theta}{4}d\theta=\int_0^{4\pi}\left(1-\cos^2\frac{\theta}{4}\right)\sin\frac{\theta}{4}d\theta.$

$\cos\dfrac{\theta}{4} = t$ とおくと $-\left(\sin\dfrac{\theta}{4}\right)\cdot\dfrac{1}{4}d\theta = dt, \sin\theta d\theta = -4dt.$

θ	0	$\to$	4π
t	1	$\to$	-1

$l = \displaystyle\int_1^{-1}(1-t^2)(-4dt) = 8\int_0^1(1-t^2)dt = 8\left[t - \frac{t^3}{3}\right]_0^1 = 8\left(1-\frac{1}{3}\right) = \frac{16}{3}.$

249. (1) 与式 $= \displaystyle\lim_{\varepsilon\to+0}\int_\varepsilon^8 \frac{dx}{\sqrt[3]{x}} = \lim_{\varepsilon\to+0}\left[\frac{x^{\frac{2}{3}}}{\frac{2}{3}}\right]_\varepsilon^8 = \lim_{\varepsilon\to+0}\left[\frac{3}{2}\sqrt[3]{x^2}\right]_\varepsilon^8 = \lim_{\varepsilon\to+0}\frac{3}{2}\left(\sqrt[3]{8^2} - \sqrt[3]{\varepsilon^2}\right) = 6.$

(与式 $= \displaystyle\left[\frac{3}{2}\sqrt[3]{x^2}\right]_0^8 = \frac{3}{2}\sqrt[3]{8^2} = 6.$ でもよい)

(2) 与式 $= \displaystyle\lim_{b\to\infty}\int_1^b \frac{dx}{x^4} = \lim_{b\to\infty}\left[\frac{x^{-3}}{-3}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{3b^3} + \frac{1}{3}\right) = \frac{1}{3}.$ (与式 $= \displaystyle\left[-\frac{1}{3x^3}\right]_1^\infty = \frac{1}{3}.$ でもよい)

(3) 与式 $= \displaystyle\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right]_0^\infty = \frac{1}{\sqrt{3}}\tan^{-1}\infty = \frac{1}{\sqrt{3}}\cdot\frac{\pi}{2} = \frac{\pi}{2\sqrt{3}}.$ (公式 $\displaystyle\int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$ を用いた)

250. (1) $\displaystyle\int_0^2 v(t)dt = \int_0^2 e^{-t}dt = \left[-e^{-t}\right]_0^2 = -e^{-2} + 1 = 1 - \frac{1}{e^2}.$

(2) $\displaystyle\int_1^2 v(t)dt = \int_1^2 2\sin\pi t dt = \left[\frac{2}{\pi}(-\cos\pi t)\right]_1^2 = -\frac{2\cos2\pi}{\pi} + \frac{2\cos\pi}{\pi} = -\frac{4}{\pi}.$ よって $\left|-\dfrac{4}{\pi}\right| = \dfrac{4}{\pi}.$