大日本図書　新微分積分 II 問題集　2 章§ 1 BASIC 詳しい解答

p.12. 2 章§ 1. 偏微分法　BASIC

43. 平面 $ax+by+cz+d=0$ の法線ベクトルの 1 つは (a,b,c)

(1) $3x-y-z+2=0$ より $(3,-1,-1)$.　(2) $2x+3y+z-1=0$ より $(2,3,1)$.

44. 解答参照

45. (1) $z_x=8x-3y, z_y=-3x+12y$.　(2) $z_x=10xy+3y^3, z_y=5x^2+9xy^2$.

(3) $z_x=\dfrac{1}{2}(2x^2y+3xy^2)^{-\frac{1}{2}}(2x^2y+3xy^2)_x=\dfrac{4xy+3y^2}{2\sqrt{2x^2y+3xy^2}}$,

$z_y=\dfrac{1}{2}(2x^2y+3xy^2)^{-\frac{1}{2}}(2x^2y+3xy^2)_y=\dfrac{2x^2+6xy}{2\sqrt{2x^2y+3xy^2}}=\dfrac{x^2+3xy}{\sqrt{2x^2y+3xy^2}}$.

(4) $z_x=e^{xy}(xy)_x=ye^{xy}, z_y=e^{xy}(xy)_y=xe^{xy}$.

(5) $z_x=e^{3x}(3x)'\tan 2y=3e^{3x}\tan 2y, z_y=e^{3x}\dfrac{(2y)'}{\cos^2 2y}=\dfrac{2e^{3x}}{\cos^2 2y}$.

(6) $z_x=(\cos 2x)(2x)'\log 3y=2\cos 2x\log 3y, z_y=(\sin 2x)\cdot\dfrac{(3y)'}{3y}=\dfrac{3\sin 2x}{3y}=\dfrac{\sin 2x}{y}$.

(7) $z_x=e^{2x+y}(2x+y)_x\cos(x-y)+e^{2x+y}\{-\sin(x-y)\}(x-y)_x=e^{2x+y}\{2\cos(x-y)-\sin(x-y)\}$,

$z_y=e^{2x+y}(2x+y)_y\cos(x-y)+e^{2x+y}\{-\sin(x-y)\}(x-y)_y=e^{2x+y}\{\cos(x-y)+\sin(x-y)\}$.

(8) $z_x=\log(2x+5y)+(x+3y)\cdot\dfrac{(2x+5y)_x}{2x+5y}=\log(2x+5y)+\dfrac{2(x+3y)}{2x+5y}$,

$z_y=3\log(2x+5y)+(x+3y)\cdot\dfrac{(2x+5y)_y}{2x+5y}=3\log(2x+5y)+\dfrac{5(x+3y)}{2x+5y}$.

(9) $z_x=\dfrac{(x+2y)_x(3x-2y)-(x+2y)(3x-2y)_x}{(3x-2y)^2}=\dfrac{3x-2y-3(x+2y)}{(3x-2y)^2}=-\dfrac{8y}{(3x-2y)^2}$,

$z_y=\dfrac{(x+2y)_y(3x-2y)-(x+2y)(3x-2y)_y}{(3x-2y)^2}=\dfrac{2(3x-2y)+2(x+2y)}{(3x-2y)^2}=\dfrac{8x}{(3x-2y)^2}$.

(10) $z_x=\dfrac{(\sin x-\cos y)_x(\sin x+\cos y)-(\sin x-\cos y)(\sin x+\cos y)_x}{(\sin x+\cos y)^2}=\dfrac{\cos x(\sin x+\cos y)-\cos x(\sin x-\cos y)}{(\sin x+\cos y)^2}$

$=\dfrac{2\cos x\cos y}{(\sin x+\cos y)^2}$,

$z_y=\dfrac{(\sin x-\cos y)_y(\sin x+\cos y)-(\sin x-\cos y)(\sin x+\cos y)_y}{(\sin x+\cos y)^2}=\dfrac{\sin y(\sin x+\cos y)+\sin y(\sin x-\cos y)}{(\sin x+\cos y)^2}$

$=\dfrac{2\sin x\sin y}{(\sin x+\cos y)^2}$.

46. (1) $f_x=4x-y, z_y=-x+6y$. $f_x(1,2)=2, f_y(1,2)=11$.

(2) $f_x=e^{x^2y}(x^2y)_x=2xye^{x^2y}, f_y=e^{x^2y}(x^2y)_y=x^2e^{x^2y}$. $f_x(1,2)=4e^2, f_y(1,2)=e^2$.

(3) $f_x=\dfrac{(x+y^2)_x}{x+y^2}=\dfrac{1}{x+y^2}, f_y=\dfrac{(x+y^2)_y}{x+y^2}=\dfrac{2y}{x+y^2}$. $f_x(1,2)=\dfrac{1}{5}, f_y(1,2)=\dfrac{4}{5}$.

(4) $f_x=\dfrac{1}{2}(xy^2+1)^{-\frac{1}{2}}(xy^2+1)_x=\dfrac{y^2}{2\sqrt{xy^2+1}}, f_y=\dfrac{1}{2}(xy^2+1)^{-\frac{1}{2}}(xy^2+1)_y=\dfrac{2xy}{2\sqrt{xy^2+1}}=\dfrac{xy}{\sqrt{xy^2+1}}$.

$f_x(1,2)=\dfrac{4}{2\sqrt{5}}=\dfrac{2}{\sqrt{5}}=\dfrac{2\sqrt{5}}{5}, f_y(1,2)=\dfrac{2}{\sqrt{5}}=\dfrac{2\sqrt{5}}{5}$.

47. (1) $f_x=2y+z, f_y=2x+3z, f_z=x+3y$. $f_x(1,2,1)=5, f_y(1,2,1)=5, f_z(1,2,1)=7$.

(2) $f_x=3(2x-3y+2z)^2(2x-3y+2z)_x=6(2x-3y+2z)^2, f_y=3(2x-3y+2z)^2(2x-3y+2z)_y$

$=-9(2x-3y+2z)^2, f_z=3(2x-3y+2z)^2(2x-3y+2z)_z=6(2x-3y+2z)^2$.

$f_x(1,2,1)=24, f_y(1,2,1)=-36, f_z(1,2,1)=24$.

(3) $f_x=\dfrac{z}{y}, f_y=xz\cdot(-1)y^{-2}=-\dfrac{xz}{y^2}, f_z=\dfrac{x}{y}$. $f_x(1,2,1)=\dfrac{1}{2}, f_y(1,2,1)=-\dfrac{1}{4}, f_z(1,2,1)=\dfrac{1}{2}$.

(4) $f_x=e^{x^2+y^2+z^2}(x^2+y^2+z^2)_x=2xe^{x^2+y^2+z^2}, f_y=e^{x^2+y^2+z^2}(x^2+y^2+z^2)_y=2ye^{x^2+y^2+z^2}$,

$f_z=e^{x^2+y^2+z^2}(x^2+y^2+z^2)_z=2ze^{x^2+y^2+z^2}$. $f_x(1,2,1)=2e^6, f_y(1,2,1)=4e^6, f_z(1,2,1)=2e^6$.

48. $z=f(x,y)$ の全微分 $dz=f_xdx+f_ydy$

(1) $z_x=6x^2y^2-4y^3, z_y=4x^3y-12xy^2$. よって $dz=(6x^2y^2-4y^3)dx+(4x^3y-12xy^2)dy$.

(2) $z_x = 4\sqrt{3y+2}, z_y = (4x+1)\dfrac{1}{2}(3y+2)^{-\frac{1}{2}}(3y+2)' = \dfrac{3(4x+1)}{2\sqrt{3y+2}}$. よって $dz = 4\sqrt{3y+2}dx + \dfrac{3(4x+1)}{2\sqrt{3y+2}}dy$.

(3) $z_x = 4(3x+5y)^3(3x+5y)_x = 12(3x+5y)^3, z_y = 4(3x+5y)^3(3x+5y)_y = 20(3x+5y)^3$.

よって $dz = 12(3x+5y)^3dx + 20(3x+5y)^3dy$.

(4) $z_x = \dfrac{(x^2+y^3)_x}{\cos^2(x^2+y^3)} = \dfrac{2x}{\cos^2(x^2+y^3)}, z_y = \dfrac{(x^2+y^3)_y}{\cos^2(x^2+y^3)} = \dfrac{3y^2}{\cos^2(x^2+y^3)}$.

よって $dz = \dfrac{2x}{\cos^2(x^2+y^3)}dx + \dfrac{3y^2}{\cos^2(x^2+y^3)}dy$.

(5) $z_x = 2e^{x+3y}+(2x+y)e^{x+3y}(x+3y)_x = (2x+y+2)e^{x+3y}, z_y = e^{x+3y}+(2x+y)e^{x+3y}(x+3y)_y = (6x+3y+1)e^{x+3y}$.

よって $dz = (2x+y+2)e^{x+3y}dx + (6x+3y+1)e^{x+3y}dy$.

(6) $z_x = \dfrac{(2x-3y)_x(x^2+y^2)-(2x-3y)(x^2+y^2)_x}{(x^2+y^2)^2} = \dfrac{2(x^2+y^2)-2x(2x-3y)}{(x^2+y^2)^2} = \dfrac{-2x^2+6xy+2y^2}{(x^2+y^2)^2}$,

$z_y = \dfrac{(2x-3y)_y(x^2+y^2)-(2x-3y)(x^2+y^2)_y}{(x^2+y^2)^2} = \dfrac{-3(x^2+y^2)-2y(2x-3y)}{(x^2+y^2)^2} = \dfrac{-3x^2-4xy+3y^2}{(x^2+y^2)^2}$.

よって $dz = \dfrac{-2x^2+6xy+2y^2}{(x^2+y^2)^2}dx + \dfrac{-3x^2-4xy+3y^2}{(x^2+y^2)^2}dy$.

49. $S = 2\pi x^2 + 2\pi xy$. $S_x = 4\pi x + 2\pi y = 2\pi(2x+y), S_y = 2\pi x$ より $\Delta S \fallingdotseq S_x\Delta x + S_y\Delta y = 2\pi(2x+y)\Delta x + 2\pi x\Delta y$.

50. 曲面 $z = f(x,y)$ 上の点 $(a,b,f(a,b))$ における接平面の方程式は $z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$.

(1) $z_x = 2x, z_y = 4y$. $(x,y) = (1,1)$ のとき $z_x = 2(= f_x(1,1)), z_y = 4(= f_y(1,1))$. ($f(1,1) = 3$ だから)

求める接平面の方程式は $z - 3 = 2(x-1) + 4(y-1)$, すなわち $2x + 4y - z = 3$.

(2) $z_x = \dfrac{1}{2}(5-x^2y^2)^{-\frac{1}{2}}(5-x^2y^2)_x = \dfrac{-2xy^2}{2\sqrt{5-x^2y^2}} = -\dfrac{xy^2}{\sqrt{5-x^2y^2}}$,

$z_y = \dfrac{1}{2}(5-x^2y^2)^{-\frac{1}{2}}(5-x^2y^2)_y = \dfrac{-2x^2y}{2\sqrt{5-x^2y^2}} = -\dfrac{x^2y}{\sqrt{5-x^2y^2}}$.

$(x,y) = (1,2)$ のとき $z_x = -\dfrac{4}{\sqrt{5-4}} = -4(= f_x(1,2)), z_y = -\dfrac{2}{\sqrt{5-4}} = -2(= f_y(1,2))$. ($f(1,2) = 1$ だから)

求める接平面の方程式は $z - 1 = -4(x-1) - 2(y-2)$, すなわち $4x + 2y + z = 9$.

(3) $z_x = \{\cos(x-y^2)\}(x-y^2)_x = \cos(x-y^2), z_y = \{\cos(x-y^2)\}(x-y^2)_y = -2y\cos(x-y^2)$.

$(x,y) = (1,1)$ のとき $z_x = \cos 0 = 1(= f_x(1,1)), z_y = -2\cos 0 = -2(= f_y(1,1)), z(= f(1,1)) = \sin 0 = 0$.

求める接平面の方程式は $z - 0 = (x-1) - 2(y-1)$, すなわち $x - 2y - z = -1$.

(4) $z_x = \dfrac{(x^2+y^2)_x}{x^2+y^2} = \dfrac{2x}{x^2+y^2}, z_y = \dfrac{(x^2+y^2)_y}{x^2+y^2} = \dfrac{2y}{x^2+y^2}$.

$(x,y) = (1,0)$ のとき $z_x = 2(= f_x(1,0)), z_y = 0(= f_y(1,0)), z(= f(1,0)) = \log 1 = 0$.

求める接平面の方程式は $z - 0 = 2(x-1) + 0(y-0)$, すなわち $2x - z = 2$.

51. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$

(1) $\dfrac{dx}{dt} = e^t + te^t = (1+t)e^t, \dfrac{dy}{dt} = \dfrac{1}{t}$. よって $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = (1+t)e^t\dfrac{\partial z}{\partial x} + \dfrac{1}{t}\dfrac{\partial z}{\partial y}$.

(2) $\dfrac{dx}{dt} = \dfrac{(2t+1)-t\cdot 2}{(2t+1)^2} = \dfrac{1}{(2t+1)^2}, \dfrac{dy}{dt} = \dfrac{(2t+1)-(t+1)\cdot 2}{(2t+1)^2} = -\dfrac{1}{(2t+1)^2}$.

よって $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = \dfrac{1}{(2t+1)^2}\dfrac{\partial z}{\partial x} - \dfrac{1}{(2t+1)^2}\dfrac{\partial z}{\partial y}$.

(3) $\dfrac{dx}{dt} = \cos t - \sin t, \dfrac{dy}{dt} = \cos^2 t - \sin^2 t = \cos 2t$. よって $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = (\cos t - \sin t)\dfrac{\partial z}{\partial x} + \cos 2t\dfrac{\partial z}{\partial y}$.

(4) $\dfrac{dx}{dt} = -\dfrac{1}{2}(t+1)^{-\frac{3}{2}}(1+t)' = -\dfrac{1}{2\sqrt{(1+t)^3}}, \dfrac{dy}{dt} = \dfrac{1}{2}(t+1)^{-\frac{1}{2}}(1+t)' = \dfrac{1}{2\sqrt{1+t}}$.

よって $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = -\dfrac{1}{2\sqrt{(1+t)^3}}\dfrac{\partial z}{\partial x} + \dfrac{1}{2\sqrt{1+t}}\dfrac{\partial z}{\partial y}$.

52. (1) $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = -\dfrac{y}{x^2}(e^t - e^{-t}) + \dfrac{1}{x}(e^t + e^{-t}) = -\dfrac{(e^t-e^{-t})^2}{(e^t+e^{-t})^2} + \dfrac{e^t+e^{-t}}{e^t+e^{-t}} = \dfrac{-(e^t-e^{-t})^2+(e^t+e^{-t})^2}{(e^t+e^{-t})^2}$

$= \dfrac{-e^{2t}+2e^te^{-t}-e^{-2t}+e^{2t}+2e^te^{-t}+e^{-2t}}{(e^t+e^{-t})^2} = \dfrac{4}{(e^t+e^{-t})^2}$. ($e^te^{-t} = e^t\dfrac{1}{e^t} = 1$ に注意)

(2) $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = e^{x-y}(x-y)_x \cos t + e^{x-y}(x-y)_y(-\sin t) = e^{x-y}(\cos t + \sin t) = e^{\sin t - \cos t}(\sin t + \cos t).$

(3) $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = \dfrac{(x+y)_x}{x+y}\dfrac{1}{2}(t+1)^{-\frac{1}{2}}(t+1)' + \dfrac{(x+y)_y}{x+y}\dfrac{1}{2}(t-1)^{-\frac{1}{2}}(t-1)'$

$= \dfrac{1}{2(x+y)\sqrt{t+1}} + \dfrac{1}{2(x+y)\sqrt{t-1}} = \dfrac{1}{2(\sqrt{t+1}+\sqrt{t-1})\sqrt{t+1}} + \dfrac{1}{2(\sqrt{t+1}+\sqrt{t-1})\sqrt{t-1}}$

$= \dfrac{\sqrt{t-1}+\sqrt{t+1}}{2(\sqrt{t+1}+\sqrt{t-1})\sqrt{t+1}\sqrt{t-1}} = \dfrac{1}{2\sqrt{t+1}\sqrt{t-1}} = \dfrac{1}{2\sqrt{t^2-1}}.$

(4) $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = \{\cos(x+2y)\}(x+2y)_x\dfrac{1}{t} + \{\cos(x+2y)\}(x+2y)_y(-2)t^{-2}$

$= \dfrac{\cos(x+2y)}{t} + \dfrac{-4\cos(x+2y)}{t^2} = \dfrac{(t-4)\cos(x+2y)}{t^2} = \dfrac{t-4}{t^2}\cos\left(\log t + \dfrac{4}{t}\right).$

53. $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u}, \dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v}$ （$z_u = z_x x_u + z_y y_u, z_v = z_x x_v + z_y y_v$ とも表す）

(1) $z_u = z_x x_u + z_y y_u = z_x 4uv^3 + z_y \cdot 1 = 4uv^3 z_x + z_y,\quad z_v = z_x x_v + z_y y_v = z_x 6u^2v^2 + z_y \cdot 3 = 6u^2v^2 z_x + 3z_y.$

(2) $z_u = z_x x_u + z_y y_u = z_x 2u + z_y \dfrac{1}{v} = 2u z_x + \dfrac{1}{v} z_y,\quad z_v = z_x x_v + z_y y_v = z_x 2v + z_y\left(-\dfrac{u}{v^2}\right) = 2v z_x - \dfrac{u}{v^2} z_y.$

(3) $z_u = z_x x_u + z_y y_u = z_x \dfrac{(\frac{v}{u})_u}{\cos^2 \frac{v}{u}} + z_y\{-\sin(u+v)\}(u+v)_u = -\dfrac{v}{u^2\cos^2\frac{v}{u}} z_x - \sin(u+v) z_y,$

$z_v = z_x x_v + z_y y_v = z_x \dfrac{(\frac{v}{u})_v}{\cos^2 \frac{v}{u}} + z_y - \sin(u+v)\}(u+v)_v = \dfrac{1}{u\cos^2\frac{v}{u}} z_x - \sin(u+v) z_y.$

(4) $z_u = z_x x_u + z_y y_u = z_x \log v + z_y e^u v = (\log v) z_x + e^u v z_y,\quad z_v = z_x x_v + z_y y_v = z_x \dfrac{u}{v} + z_y e^u = \dfrac{u}{v} z_x + e^u z_y.$

54. (1) $z_u = z_x x_u + z_y y_u = 2xy \cdot 1 + x^2 v = 2(u+v)uv + (u+v)^2 v = v(u+v)(3u+v),$

$z_v = z_x x_v + z_y y_v = 2xy \cdot 1 + x^2 u = 2(u+v)uv + (u+v)^2 u = u(u+v)(u+3v).$

(2) $z_u = z_x x_u + z_y y_u = \dfrac{1}{y} \cdot 2 - \dfrac{x}{y^2} \cdot 3 = \dfrac{2y-3x}{y^2} = \dfrac{2(3u-2v)-3(2u+3v)}{(3u-2v)^2} = -\dfrac{13v}{(3u-2v)},$

$z_v = z_x x_v + z_y y_v = \dfrac{1}{y} \cdot 3 - \dfrac{x}{y^2} \cdot (-2) = \dfrac{3y+2x}{y^2} = \dfrac{3(3u-2v)+2(2u+3v)}{(3u-2v)^2} = \dfrac{13u}{(3u-2v)}.$

(3) $z_u = z_x x_u + z_y y_u = 2 \cdot \dfrac{1}{2}(x+y)^{-\frac{1}{2}}(x+y)_x \cos(2u+v)(2u+v)_u + 2 \cdot \dfrac{1}{2}(x+y)^{-\frac{1}{2}}(x+y)_y\{-\sin(u-2v)\}(u-2v)_u$

$= \dfrac{2\cos(2u+v) - \sin(u-2v)}{\sqrt{x+y}} = \dfrac{2\cos(2u+v) - \sin(u-2v)}{\sqrt{\sin(2u+v)+\cos(u-2v)}},$

$z_v = z_x x_v + z_y y_v = 2 \cdot \dfrac{1}{2}(x+y)^{-\frac{1}{2}}(x+y)_x \cos(2u+v)(2u+v)_v + 2 \cdot \dfrac{1}{2}(x+y)^{-\frac{1}{2}}(x+y)_y\{-\sin(u-2v)\}(u-2v)_v$

$= \dfrac{\cos(2u+v) + 2\sin(u-2v)}{\sqrt{x+y}} = \dfrac{\cos(2u+v) + 2\sin(u-2v)}{\sqrt{\sin(2u+v)+\cos(u-2v)}}.$

(4) $z_u = z_x x_u + z_y y_u = 2x(\log y) \cdot 2 + \dfrac{x^2}{y} \cdot v = 4x\log y + \dfrac{x^2 v}{y} = 4(2u+v)\log(uv) + \dfrac{(2u+v)^2 v}{uv}$

$= 4(2u+v)\log(uv) + \dfrac{(2u+v)^2}{u},$

$z_v = z_x x_v + z_y y_v = 2x(\log y) \cdot 1 + \dfrac{x^2}{y} \cdot u = 2x\log y + \dfrac{x^2 u}{y} = 2(2u+v)\log(uv) + \dfrac{(2u+v)^2 u}{uv}$

$= 2(2u+v)\log(uv) + \dfrac{(2u+v)^2}{v}.$

p.14. 2 章§ 1. 偏微分法　CHECK

55. (1) z が $\sqrt{x^2+y^2}$ の関数だから z 軸を中心とする回転面である．曲面と zx 平面 $(y=0)$ との交線は $z = e^{\sqrt{x^2}} = e^{|x|}$ だから指数関数 $z = e^x$ の $x \geqq 0$ の部分を z 軸のまわりに回転してできる回転面．(グラフは解答参照)．

(2) 同様に z 軸を中心とする回転面である．曲面と zx 平面 $(y=0)$ との交線は $z = 4 - x^2$ だから放物線 $z = 4 - x^2$ を z 軸のまわりに回転してできる回転面．(グラフは解答参照)．

56. (1) $z_x = 10x - y, z_y = -x + 4y$

(2) $z_x = \{\cos(x-y)\}(x-y)_x = \cos(x-y), z_y = \{\cos(x-y)\}(x-y)_y = -\cos(x-y).$

(3) $z_x = \dfrac{(x+y) - x(x+y)_x}{(x+y)^2} = \dfrac{(x+y)-x}{(x+y)^2} = \dfrac{y}{(x+y)^2},$

$z_y = \dfrac{(x)_y(x+y) - x(x+y)_y}{(x+y)^2} = \dfrac{0-x}{(x+y)^2} = -\dfrac{x}{(x+y)^2}.$

(4) $z = (x^2-y^2)^{-\frac{1}{2}}$. $z_x = -\dfrac{1}{2}(x^2-y^2)^{-\frac{3}{2}}(x^2-y^2)_x = -\dfrac{1}{2}(x^2-y^2)^{-\frac{3}{2}}2x = -\dfrac{x}{\sqrt{(x^2-y^2)^3}},$

$z_y = -\dfrac{1}{2}(x^2-y^2)^{-\frac{3}{2}}(x^2-y^2)_y = -\dfrac{1}{2}(x^2-y^2)^{-\frac{3}{2}}(-2y) = \dfrac{y}{\sqrt{(x^2-y^2)^3}}.$

57. (1) $f_x = 4xy+3y^2, f_y = 2x^2+6xy$. $f_x(1,2) = 20, f_y(1,2) = 14$.

(2) $f_x = ye^{xy} + xye^{xy}(xy)_x = y(1+xy)e^{xy}, f_y = xe^{xy} + xye^{xy}(xy)_y = x(1+xy)e^{xy}$. $f_x(1,2) = 6e^2, f_y(1,2) = 3e^2$.

(3) $f_x = \dfrac{(x^2+y)_x}{x+y^2} = \dfrac{2x}{x^2+y}, f_y = \dfrac{(x^2+y)_y}{x+y^2} = \dfrac{1}{x^2+y}$. $f_x(1,2) = \dfrac{2}{3}, f_y(1,2) = \dfrac{1}{3}$.

(4) $f_x = \dfrac{1}{2}(3x^2+xy+y^2)^{-\frac{1}{2}}(3x^2+xy+y^2)_x = \dfrac{6x+y}{2\sqrt{3x^2+xy+y^2}}, f_y = \dfrac{1}{2}(3x^2+xy+y^2)^{-\frac{1}{2}}(3x^2+xy+y^2)_y$

$= \dfrac{x+2y}{2\sqrt{3x^2+xy+y^2}}$. $f_x(1,2) = \dfrac{4}{3}, f_y(1,2) = \dfrac{5}{6}$.

58. (1) $z_x = 8xy+5y^3, z_y = 4x^2+15xy^2$. よって $dz = (8xy+5y^3)dx + (4x^2+15xy^2)dy$.

(2) $z_x = (\cos\sqrt{3x+2y})\cdot\dfrac{1}{2}(3x+2y)^{-\frac{1}{2}}(3x+2y)_x = \dfrac{3\cos\sqrt{3x+2y}}{2\sqrt{3x+2y}}, z_y = (\cos\sqrt{3x+2y})\cdot\dfrac{1}{2}(3x+2y)^{-\frac{1}{2}}(3x+2y)_y$

$= \dfrac{\cos\sqrt{3x+2y}}{\sqrt{3x+2y}}$. よって $dz = \dfrac{3\cos\sqrt{3x+2y}}{2\sqrt{3x+2y}}dx + \dfrac{\cos\sqrt{3x+2y}}{\sqrt{3x+2y}}dy$.

59. $V = xy(x+y) = x^2y+xy^2$. よって $V_x = 2xy+y^2, V_y = x^2+2xy$ だから $\Delta V \fallingdotseq (2xy+y^2)\Delta x + (x^2+2xy)\Delta y$.

60. (1) $z_x = \dfrac{1}{2}\left(1-\dfrac{x^2}{4}-\dfrac{y^2}{9}\right)^{-\frac{1}{2}}\left(1-\dfrac{x^2}{4}-\dfrac{y^2}{9}\right)_x = -\dfrac{x}{4\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}},$

$z_y = \dfrac{1}{2}\left(1-\dfrac{x^2}{4}-\dfrac{y^2}{9}\right)^{-\frac{1}{2}}\left(1-\dfrac{x^2}{4}-\dfrac{y^2}{9}\right)_y = -\dfrac{y}{9\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}}.$

$\left(\dfrac{4}{3}, 2, \dfrac{1}{3}\right)$ のとき $z_x = -\dfrac{\frac{4}{3}}{4\cdot\frac{1}{3}} = -1, z_y = -\dfrac{2}{9\cdot\frac{1}{3}} = -\dfrac{2}{3}$. よって $z-\dfrac{1}{3} = -\left(x-\dfrac{4}{3}\right) - \dfrac{2}{3}(y-2)$ より

$3x+2y+3z = 9$.

(2) $z = (x^2+y^2+y)^{-1}, z_x = -(x^2+y^2+y)^{-2}(x^2+y^2+y)_x = -\dfrac{2x}{(x^2+y^2+y)^2},$

$z_y = -(x^2+y^2+y)^{-2}(x^2+y^2+y)_y = -\dfrac{2y+1}{(x^2+y^2+y)^2}.$

$x=1, y=0$ のとき $z = 1, z_x = -2, z_y = -1$. よって $z-1 = -2(x-1)-(y-0)$ より $2x+y+z = 3$.

61. (1) $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = \{\cos(3x+2y)\}(3x+2y)_x(-t^{-2}) + \{\cos(3x+2y)\}(3x+2y)_y\left(\dfrac{1}{2}t^{-\frac{1}{2}}\right)$

$= -\dfrac{3}{t^2}\cos(3x+2y) + \dfrac{1}{\sqrt{t}}\cos(3x+2y)$. $x = \dfrac{1}{t}, y = \sqrt{t}$ より $\dfrac{dz}{dt} = \left(-\dfrac{3}{t^2}+\dfrac{1}{\sqrt{t}}\right)\cos\left(\dfrac{3}{t}+2\sqrt{t}\right)$.

(2) $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = \dfrac{(2x^2+xy+5y^2)_x}{2x^2+xy+5y^2}(-\sin t) + \dfrac{(2x^2+xy+5y^2)_x}{2x^2+xy+5y^2}\cos t = \dfrac{-(4x+y)\sin t + (x+10y)\cos t}{2x^2+xy+5y^2}.$

$x = \cos t, y = \sin t$ より $\dfrac{dz}{dt} = \dfrac{-(4\cos t+\sin t)\sin t + (\cos t+10\sin t)\cos t}{2\cos^2 t+\cos t\sin t+5\sin^2 t} = \dfrac{\cos^2 t+6\sin t\cos t+\sin^2 t}{2\cos^2 t+\sin t\cos t+5\sin^2 t}.$

62. (1) $z_u = z_x x_u + z_y y_u = -(x+y)^{-2}(x+y)_x\cdot 1 - (x+y)^{-2}(x+y)_y\cdot v = \dfrac{-1-v}{(x+y)^2},$

$z_v = z_x x_v + z_y y_v = -(x+y)^{-2}(x+y)_x\cdot 1 - (x+y)^{-2}(x+y)_y\cdot u = \dfrac{-1-u}{(x+y)^2}.$

$x = u+v, y = uv$ より $z_u = -\dfrac{1+v}{(u+v+uv)^2}, z_v = -\dfrac{1+u}{(u+v+uv)^2}.$

(2) $z_u = z_x x_u + z_y y_u = \dfrac{(2x+3y)_x}{2x+3y}\{\cos(u+v)\}(u+v)_u + \dfrac{(2x+3y)_y}{2x+3y}\{-\sin(u-v)\}(u-v)_u = \dfrac{\cos(u+v)-3\sin(u+v)}{2x+3y},$

$z_v = z_x x_v + z_y y_v = \dfrac{(2x+3y)_x}{2x+3y}\{\cos(u+v)\}(u+v)_v + \dfrac{(2x+3y)_y}{2x+3y}\{-\sin(u-v)\}(u-v)_v = \dfrac{\cos(u+v)+3\sin(u+v)}{2x+3y}.$

$x = \sin(u+v), y = \cos(u-v)$ より $z_u = \dfrac{\cos(u+v)-3\sin(u+v)}{2\sin(u+v)+3\cos(u-v)}, z_v = \dfrac{\cos(u+v)+3\sin(u+v)}{2\sin(u+v)+3\cos(u-v)}.$