p.27. 3 章§ 1. 2 重積分　BASIC

108. 半球面 $x^2+y^2+z^2=9, z \geqq 0$ は $z=\sqrt{9-x^2-y^2}$ だから $D=\{(x,y) \mid x^2+y^2 \leqq 4\}$ とすると
$V=\iint_D \sqrt{9-x^2-y^2}dxdy.$

109. $2 \leqq x \leqq 3, 0 \leqq y \leqq 1$ より $p=2, q=3, r=0, s=1.$ $\iint_D (x^2y-y^3)dxdy=\int_2^3 \left\{\int_0^1 (x^2y-y^3)dy\right\}dx$
$=\int_2^3 \left[\frac{x^2y^2}{2}-\frac{y^4}{4}\right]_0^1 dx=\int_2^3 \left(\frac{x^2}{2}-\frac{1}{4}\right)dx=\left[\frac{x^3}{6}-\frac{1}{4}x\right]_2^3=\left(\frac{9}{2}-\frac{3}{4}\right)-\left(\frac{4}{3}-\frac{1}{2}\right)=\frac{35}{12}.$

110. (1) 与式 $=\int_0^1 \left\{\int_1^3 (xy^2+y)dy\right\}dx=\int_0^1 \left[\frac{xy^3}{3}+\frac{y^2}{2}\right]_1^3 dx=\int_0^1 \left(9x+\frac{9}{2}-\frac{x}{3}-\frac{1}{2}\right)dx=\int_0^1 \left(\frac{26}{3}x+4\right)dx$
$=\left[\frac{13}{3}x^2+4x\right]_0^1=\frac{13}{3}+4=\frac{25}{3}.$

(2) 与式 $=\int_0^1 \left\{\int_1^2 e^{2x+y}dy\right\}dx=\int_0^1 [e^{2x+y}]_1^2 dx=\int_0^1 (e^{2x+2}-e^{2x+1})dx=\left[\frac{1}{2}e^{2x+2}-\frac{1}{2}e^{2x+1}\right]_0^1$
$=\frac{1}{2}\{(e^4-e^3)-(e^2-e)\}=\frac{1}{2}e(e+1)(e-1)^2.$

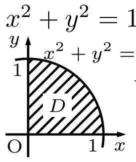
(3) 与式 $=\int_0^{\frac{\pi}{2}} \left\{\int_0^{\frac{\pi}{2}} \sin(x-y)dy\right\}dx=\int_0^{\frac{\pi}{2}} [\cos(x-y)]_0^{\frac{\pi}{2}} dx=\int_0^{\frac{\pi}{2}} \left\{\cos\left(x-\frac{\pi}{2}\right)-\cos x\right\}dx$
$=\left[\sin\left(x-\frac{\pi}{2}\right)-\sin x\right]_0^{\frac{\pi}{2}}=\left(\sin 0-\sin\frac{\pi}{2}\right)-\left\{\sin\left(-\frac{\pi}{2}\right)-\sin 0\right\}=0-1-(-1-0)=0.$

111. (1) 与式 $=\int_0^1 \left\{\int_0^{\sqrt{x}} xydy\right\}dx=\int_0^1 \left[\frac{xy^2}{2}\right]_0^{\sqrt{x}} dx=\int_0^1 \frac{x^2}{2}dx=\left[\frac{x^3}{6}\right]_0^1=\frac{1}{6}.$

(2) 与式 $=\int_0^1 \left\{\int_y^{2y} (x-y)dx\right\}dy=\int_0^1 \left[\frac{x^2}{2}-xy\right]_y^{2y} dy=\int_0^1 \left\{(2y^2-2y^2)-\left(\frac{y^2}{2}-y^2\right)\right\}dy=\int_0^1 \frac{y^2}{2}dy$
$=\left[\frac{y^3}{6}\right]_0^1=\frac{1}{6}.$

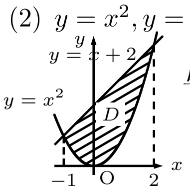
(3) 与式 $=\int_1^3 \left\{\int_1^{x^2} \frac{x}{y^2}dy\right\}dx=\int_1^3 \left[\frac{xy^{-1}}{-1}\right]_1^{x^2} dx=\int_1^3 \left(-\frac{1}{x}+x\right)dx=\left[-\log|x|+\frac{x^2}{2}\right]_1^3$
$=-\log 3+\frac{9}{2}+\log 1-\frac{1}{2}=4-\log 3.$

112. (1) $x^2+y^2=1 \Rightarrow y=\sqrt{1-x^2}$ より $D=\{(x,y) \mid 0 \leqq x \leqq 1, 0 \leqq y \leqq \sqrt{1-x^2}\}.$



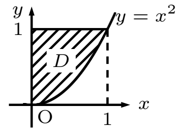
与式 $=\int_0^1 \left\{\int_0^{\sqrt{1-x^2}} x^2ydy\right\}dx=\int_0^1 \left[\frac{x^2y^2}{2}\right]_0^{\sqrt{1-x^2}} dx=\frac{1}{2}\int_0^1 x^2(1-x^2)dx=\frac{1}{2}\left[\frac{x^3}{3}-\frac{x^5}{5}\right]_0^1$
$=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)=\frac{1}{15}.$

(2) $y=x^2, y=x+2 \Rightarrow x^2-x-2=(x+1)(x-2)=0 \Rightarrow x=-1,2$ より $D=\{(x,y) \mid -1 \leqq x \leqq 2, x^2 \leqq y \leqq x+2\}.$

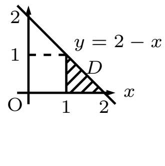


与式 $=\int_{-1}^2 \left\{\int_{x^2}^{x+2} (x-2y)dy\right\}dx=\int_{-1}^2 [xy-y^2]_{x^2}^{x+2} dx=\int_{-1}^2 \{x(x+2)-(x+2)^2-(x^3-x^4)\}dx$
$=\int_{-1}^2 (x^4-x^3-2x-4)dx=\left[\frac{x^5}{5}-\frac{x^4}{4}-x^2-4x\right]_{-1}^2=\left(\frac{32}{5}-4-4-8\right)-\left(-\frac{1}{5}-\frac{1}{4}-1+4\right)=-\frac{243}{20}.$

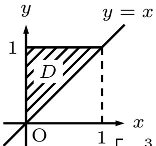
113. (1) $y=x^2 \Rightarrow x=\sqrt{y}$ より $D=\{(x,y) \mid 0 \leqq x \leqq 1, x^2 \leqq y \leqq 1\}=\{(x,y) \mid 0 \leqq y \leqq 1, 0 \leqq x \leqq \sqrt{y}\}.$
与式 $=\int_0^1 \left\{\int_0^{\sqrt{y}} f(x,y)dx\right\}dy.$



(2) $x=2-y \Rightarrow y=2-x$ より $D=\{(x,y) \mid 0 \leqq y \leqq 1, 1 \leqq x \leqq 2-y\}$
$=\{(x,y) \mid 1 \leqq x \leqq 2, 0 \leqq y \leqq 2-x\}.$ 与式 $=\int_1^2 \left\{\int_0^{2-x} f(x,y)dy\right\}dx.$



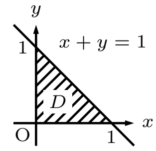
114. $D=\{(x,y)\mid 0\leqq x\leqq 1, x\leqq y\leqq 1\}=\{(x,y)\mid 0\leqq y\leqq 1, 0\leqq x\leqq y\}$



与式 $=\displaystyle\int_0^1\left\{\int_0^y\sqrt{y^2+1}dx\right\}dy=\int_0^1\sqrt{y^2+1}[x]_0^y dy=\int_0^1 y\sqrt{y^2+1}dy.$

$y^2+1=t$ とおくと $2ydy=dt\Rightarrow ydy=\dfrac{1}{2}dt.$ $\begin{array}{c|ccc} y & 0 & \to & 1 \\ \hline t & 1 & \to & 2 \end{array}.$ 与式 $=\displaystyle\int_1^2\sqrt{t}\frac{1}{2}dt=\frac{1}{2}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^2=\frac{1}{3}(2\sqrt{2}-1).$

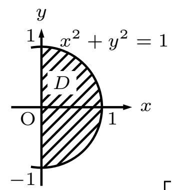
115. 3 つの座標平面 $\Leftrightarrow$ xy 平面 $(z=0)$, yz 平面 $(x=0)$, zx 平面 $(y=0)$ より 3 つの直線 $x=0, y=0, x+y=1$ で囲まれた領域 $D=\{(x,y)\mid 0\leqq x\leqq 1, 0\leqq y\leqq 1-x\}$ 上で積分.$(x+y=1\Leftrightarrow y=1-x)$



$V=\displaystyle\iint_D(x^2+2y^2)dxdy=\int_0^1\left\{\int_0^{1-x}(x^2+2y^2)dy\right\}dx=\int_0^1\left[x^2y+\frac{2}{3}y^3\right]_0^{1-x}dx$

$=\displaystyle\int_0^1\left\{x^2(1-x)+\frac{2}{3}(1-x)^3\right\}dx=\left[\frac{x^3}{3}-\frac{x^4}{4}-\frac{2}{3}\cdot\frac{(1-x)^4}{4}\right]_0^1=\left(\frac{1}{3}-\frac{1}{4}-0\right)-\left(0-0-\frac{1}{6}\right)=\frac{1}{4}.$

116. 平面 $z=2x$ が円柱を切り取るのは $z\geqq 0$, よって $x\geqq 0$ の部分だから, 円の内部の $x\geqq 0$ の部分



$D=\{(x,y)\mid x^2+y^2\leqq 1, x\geqq 0\}=\{(x,y)\mid 0\leqq x\leqq 1, -\sqrt{1-x^2}\leqq y\leqq\sqrt{1-x^2}\}$

$=\{(x,y)\mid -1\leqq y\leqq 1, 0\leqq x\leqq\sqrt{1-y^2}\}$ で積分.

$(x\geqq 0$ より $x^2+y^2=1\Leftrightarrow y=\pm\sqrt{1-x^2}\Leftrightarrow x=\sqrt{1-y^2})$

$V=\displaystyle\iint_D 2xdxdy=\int_{-1}^1\left\{\int_0^{\sqrt{1-y^2}}2xdx\right\}dy=\int_{-1}^1[x^2]_0^{\sqrt{1-y^2}}dy=\int_{-1}^1(1-y^2)dy=2\int_0^1(1-y^2)dy=2\left[y-\frac{y^3}{3}\right]_0^1$

$=2\left(1-\dfrac{1}{3}\right)=\dfrac{4}{3}.$

注：累次積分は $V=\displaystyle\int_0^1\left\{\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}2xdy\right\}dx$ でも計算できるが $V=\displaystyle\int_{-1}^1\left\{\int_0^{\sqrt{1-y^2}}2xdx\right\}dy$ の方が簡単である.

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117. (1) 与式 $=\displaystyle\int_0^2\left\{\int_{-2}^0(x^2+y^2)dy\right\}dx=\int_0^2\left[x^2y+\frac{y^3}{3}\right]_{-2}^0dx=\int_0^2\left\{0-\left(-2x^2-\frac{8}{3}\right)\right\}dx=\left[\frac{2x^3}{3}+\frac{8}{3}x\right]_0^2=\frac{32}{3}.$

(2) 与式 $=\displaystyle\int_0^1\left\{\int_1^4\sqrt{xy}dy\right\}dx=\int_0^1\sqrt{x}\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4dx=\frac{2}{3}(8-1)\int_0^1\sqrt{x}dx=\frac{14}{3}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1=\frac{28}{9}.$

(3) 与式 $=\displaystyle\int_0^1\left\{\int_0^2\frac{y}{x^2+1}dy\right\}dx=\int_0^1\left[\frac{y^2}{2(1+x^2)}\right]_0^2dx=\int_0^1\frac{2}{x^2+1}dx=2\left[\tan^{-1}x\right]_0^1=2\tan^{-1}1=\frac{\pi}{2}.$

(4) 与式 $=\displaystyle\int_1^3\left\{\int_x^{2x}\frac{y}{x}dy\right\}dx=\int_1^3\left[\frac{y^2}{2x}\right]_x^{2x}dx=\int_1^3\frac{4x^2-x^2}{2x}dx=\int_1^3\frac{3x}{2}dx=\left[\frac{3x^2}{4}\right]_1^3=\frac{3}{4}(9-1)=6.$

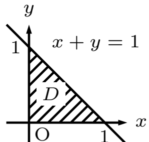
(5) 与式 $=\displaystyle\int_0^2\left\{\int_0^{\sqrt{4-y^2}}xy^2dx\right\}dy=\int_0^2\left[\frac{x^2y^2}{2}\right]_0^{\sqrt{4-y^2}}dy=\int_0^2\frac{(4-y^2)y^2}{2}dy=\int_0^2\left(2y^2-\frac{y^4}{2}\right)dy$

$=\left[\dfrac{2y^3}{3}-\dfrac{y^5}{10}\right]_0^2=\dfrac{16}{3}-\dfrac{16}{5}=\dfrac{32}{15}.$

(6) 与式 $=\displaystyle\int_1^2\left\{\int_1^{\sqrt{y}}(x^3+xy)dx\right\}dy=\int_1^2\left[\frac{x^4}{4}+\frac{x^2y}{2}\right]_1^{\sqrt{y}}dy=\int_1^2\left(\frac{y^2}{4}+\frac{y^2}{2}-\frac{1}{4}-\frac{y}{2}\right)dy$

$=\displaystyle\int_1^2\left(\frac{3y^2}{4}-\frac{y}{2}-\frac{1}{4}\right)dy=\left[\frac{y^3}{4}-\frac{y^2}{4}-\frac{y}{4}\right]_1^2=\frac{(8-4-2)-(1-1-1)}{4}=\frac{3}{4}.$

118.

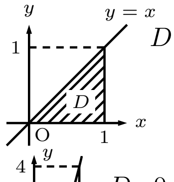


$D: 0\leqq x\leqq 1, 0\leqq y\leqq 1-x$ $(x+y=1\Rightarrow y=1-x)$

与式 $=\displaystyle\int_0^1\left\{\int_0^{1-x}x^3dy\right\}dx=\int_0^1[x^3y]_0^{1-x}dx=\int_0^1x^3(1-x)dx=\int_0^1(x^3-x^4)dx=\left[\frac{x^4}{4}-\frac{x^5}{5}\right]_0^1$

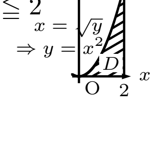
$=\dfrac{1}{4}-\dfrac{1}{5}=\dfrac{1}{20}.$

119. (1) $D: 0\leqq x\leqq 1, 0\leqq y\leqq x$

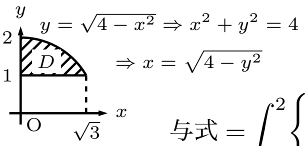


$D: 0\leqq y\leqq 1, y\leqq x\leqq 1.$ 与式 $=\displaystyle\int_0^1\left\{\int_y^1 f(x,y)dx\right\}dy.$

(2) $D: 0\leqq y\leqq 4, \sqrt{y}\leqq x\leqq 2$



$D: 0\leqq x\leqq 2, 0\leqq y\leqq x^2.$ 与式 $=\displaystyle\int_0^2\left\{\int_0^{x^2}f(x,y)dy\right\}dx.$

120. 

$D : 0 \leqq x \leqq \sqrt{3}, 1 \leqq y \leqq \sqrt{4-x^2} \Leftrightarrow D : 1 \leqq y \leqq 2, 0 \leqq x \leqq \sqrt{4-y^2}$

与式 $= \displaystyle\int_1^2 \left\{ \int_0^{\sqrt{4-y^2}} \frac{x}{\sqrt{x^2+y^2}} dx \right\} dy.$ $x^2+y^2 = t$ とおくと $2xdx = dt \Rightarrow xdx = \dfrac{1}{2}dt.$

$\displaystyle\int \frac{x}{\sqrt{x^2+y^2}} dx = \int \frac{1}{\sqrt{t}} \frac{1}{2} dt = \frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} = \sqrt{t} = \sqrt{x^2+y^2}.$ よって

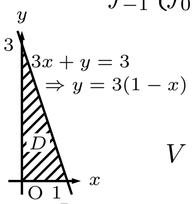
与式 $= \displaystyle\int_1^2 \left[\sqrt{x^2+y^2} \right]_0^{\sqrt{4-y^2}} dy = \int_1^2 \left(\sqrt{4-y^2+y^2} - \sqrt{y^2} \right) dy = \int_1^2 (2-y) dy = \left[2y - \frac{y^2}{2} \right]_1^2 = 4 - 2 - \left(2 - \frac{1}{2} \right)$
$= \dfrac{1}{2}.$

121. (1) $z = xy^2, z = 0$ より $x = 0, y = 0.$ よって $D : 0 \leqq x \leqq 1, 0 \leqq y \leqq 1.$ $V = \displaystyle\int_0^1 \left\{ \int_0^1 xy^2 dy \right\} dx = \int_0^1 \left[\frac{xy^3}{3} \right]_0^1 dx$

$= \displaystyle\int_0^1 \frac{x}{3} dx = \left[\frac{x^2}{6} \right]_0^1 = \frac{1}{6}.$

(2) $z = 1 - x^2, z = 0$ より $1 - x^2 = (1-x)(1+x) = 0, x = \pm 1.$ よって $D : -1 \leqq x \leqq 1, 0 \leqq y \leqq 1.$

$V = \displaystyle\int_{-1}^1 \left\{ \int_0^1 (1-x^2) dy \right\} dx = \int_{-1}^1 \left[(1-x^2)y \right]_0^1 dx = \int_{-1}^1 (1-x^2) dx = 2 \left[x - \frac{x^3}{3} \right]_0^1 = 2 \left(1 - \frac{1}{3} \right) = \frac{4}{3}.$

122. 

3 つの座標平面 $\Rightarrow$ xy 平面, yz 平面, zx 平面 $\Rightarrow z = 0, x = 0, y = 0.$

$D : 0 \leqq x \leqq 1, 0 \leqq y \leqq 3(1-x).$

$V = \displaystyle\int_0^1 \left\{ \int_0^{3(1-x)} (x+2) dy \right\} dx = \int_0^1 \left[(x+2)y \right]_0^{3(1-x)} dx = \int_0^1 3(x+2)(1-x) dx = 3 \int_0^1 (-x^2 - x + 2) dx$

$= 3 \left[-\dfrac{x^3}{3} - \dfrac{x^2}{2} + 2x \right]_0^1 = 3 \left(-\dfrac{1}{3} - \dfrac{1}{2} + 2 \right) = \dfrac{7}{2}.$