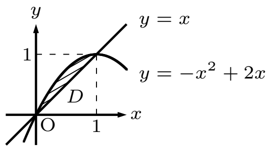
p.29. 3 章§ 1. 2 重積分　STEPUP




123. $y=x,\ y=-x^2+2x$ より $x=-x^2+2x, x^2-x=x(x-1)=0$ より $x=0,1$.

$0$ と $1$ の間の $x=\dfrac{1}{2}$ のとき $x=\dfrac{1}{2}, -x^2+2x=-\dfrac{1}{4}+1=\dfrac{3}{4}$ より

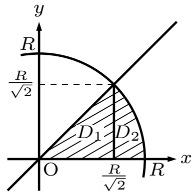
$0\leqq x\leqq 1$ で $x\leqq -x^2+2x$. よって $D: 0\leqq x\leqq 1, x\leqq y\leqq -x^2+2x$.

与式 $=\displaystyle\int_0^1\left\{\int_x^{-x^2+2x} ydy\right\}dx=\int_0^1\left[\frac{y^2}{2}\right]_x^{-x^2+2x}dx=\frac{1}{2}\int_0^1\{(-x^2+2x)^2-x^2\}dx=\frac{1}{2}\int_0^1(x^4-4x^3+3x^2)dx$

$\displaystyle =\frac{1}{2}\left[\frac{x^5}{5}-x^4+x^3\right]_0^1=\frac{1}{2}\left(\frac{1}{5}-1+1\right)=\frac{1}{10}.$

124. 与式 $=\displaystyle\int_0^1\left\{\int_{x^2}^{2-x}\frac{1}{x+1}dy\right\}dx=\int_0^1\frac{1}{x+1}\,[y]_{x^2}^{2-x}\,dx=\int_0^1\frac{2-x-x^2}{x+1}dx=\int_0^1\left(\frac{2}{x+1}-x\right)dx=\left[2\log|x+1|-\frac{x^2}{2}\right]_0^1$

$\displaystyle =2\log 2-\frac{1}{2}-2\log 1=2\log 2-\frac{1}{2}.$

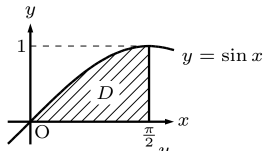



125. (1) $D: 0\leqq y\leqq \dfrac{R}{\sqrt{2}}, y\leqq x\leqq\sqrt{R^2-y^2}$ $(x=\sqrt{R^2-y^2}\Rightarrow x^2+y^2=R^2\Rightarrow y=\sqrt{R^2-x^2})$.

$0\leqq x\leqq\dfrac{R}{\sqrt{2}}$ のときと $\dfrac{R}{\sqrt{2}}\leqq x\leqq R$ のときで $y$ の上限の式が異なるから $D$ を 2 つに分ける.

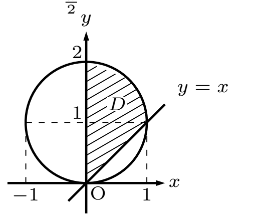
$D=D_1+D_2,\ D_1: 0\leqq x\leqq\dfrac{R}{\sqrt{2}}, 0\leqq y\leqq x,\ D_2: \dfrac{R}{\sqrt{2}}\leqq x\leqq R, 0\leqq y\leqq\sqrt{R^2-x^2}$. よって

与式 $=\displaystyle\int_0^{\frac{R}{\sqrt{2}}}\left\{\int_0^x f(x,y)dy\right\}dx+\int_{\frac{R}{\sqrt{2}}}^R\left\{\int_0^{\sqrt{R^2-x^2}}f(x,y)dy\right\}dx.$




(2) $D: 0\leqq x\leqq\dfrac{\pi}{2}, 0\leqq y\leqq\sin x$ $\left(y=\sin x\left(0\leqq x\leqq\dfrac{\pi}{2}\right)\Rightarrow x=\sin^{-1}y\right)$.

$D: 0\leqq y\leqq 1, \sin^{-1}y\leqq x\leqq\dfrac{\pi}{2}$. 与式 $=\displaystyle\int_0^1\left\{\int_{\sin^{-1}y}^{\frac{\pi}{2}}f(x,y)dx\right\}dy.$




126. (1) $x^2+y^2-2y=0$ より $x^2+(y-1)^2=1$. よって $(y-1)^2=1-x^2\Rightarrow y=1\pm\sqrt{1-x^2}$

$D: 0\leqq x\leqq 1, x\leqq y\leqq 1+\sqrt{1-x^2}$. 与式 $=\displaystyle\int_0^1\left\{\int_x^{1+\sqrt{1-x^2}}x\sqrt{y}dy\right\}dx$

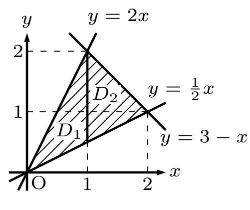
$\displaystyle =\int_0^1 x\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_x^{1+\sqrt{1-x^2}}dx=\frac{2}{3}\int_0^1\left\{x\left(1+\sqrt{1-x^2}\right)^{\frac{3}{2}}-x^{\frac{5}{2}}\right\}dx.$

$1+\sqrt{1-x^2}=t$ とおくと $1-x^2=(t-1)^2$. よって $-2xdx=2(t-1)dt\Rightarrow xdx=-(t-1)dt$.

| $x$ | $0$ | $\to$ | $1$ |
|---|---|---|---|
| $t$ | $2$ | $\to$ | $1$ |

従って $\displaystyle\int_0^1 x\left(1+\sqrt{1-x^2}\right)^{\frac{3}{2}}dx=\int_2^1 t^{\frac{3}{2}}\{-(t-1)\}dt=\int_1^2\left(t^{\frac{5}{2}}-t^{\frac{3}{2}}\right)dt=\left[\frac{2}{7}t^{\frac{7}{2}}-\frac{2}{5}t^{\frac{5}{2}}\right]_1^2$. よって

与式 $=\displaystyle\frac{2}{3}\left(\left[\frac{2}{7}t^{\frac{7}{2}}-\frac{2}{5}t^{\frac{5}{2}}\right]_1^2-\left[\frac{2}{7}x^{\frac{7}{2}}\right]_0^1\right)=\frac{2}{3}\left(\frac{2}{7}2^{\frac{7}{2}}-\frac{2}{5}2^{\frac{5}{2}}-\frac{2}{7}+\frac{2}{5}-\frac{2}{7}\right)=\frac{4}{35}(4\sqrt{2}-1).$



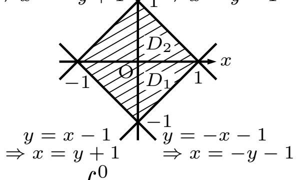

(2) 右図のように $D=D_1+D_2, D_1: 0\leqq x\leqq 1, \dfrac{1}{2}x\leqq y\leqq 2x, D_2: 1\leqq x\leqq 2, \dfrac{1}{2}x\leqq y\leqq 3-x$

与式 $=\displaystyle\int_0^1\left\{\int_{\frac{1}{2}x}^{2x}(x+y)dy\right\}dx+\int_1^2\left\{\int_{\frac{1}{2}x}^{3-x}(x+y)dy\right\}dx=\int_0^1\left[xy+\frac{y^2}{2}\right]_{\frac{1}{2}x}^{2x}dx+\int_1^2\left[xy+\frac{y^2}{2}\right]_{\frac{1}{2}x}^{3-x}dx$

$\displaystyle =\int_0^1\left(2x^2+2x^2-\frac{x^2}{2}-\frac{x^2}{8}\right)dx+\int_1^2\left\{x(3-x)+\frac{(3-x)^2}{2}-\frac{x^2}{2}-\frac{x^2}{8}\right\}dx$

$\displaystyle =\int_0^1\frac{27}{8}x^2dx+\int_1^2\left(-\frac{9}{8}x^2+\frac{9}{2}\right)dx=\left[\frac{9}{8}x^3\right]_0^1+\left[-\frac{3}{8}x^3+\frac{9}{2}x\right]_1^2=\frac{9}{8}-3+9+\frac{3}{8}-\frac{9}{2}=3.$

127.




$D=D_1+D_2, D_1: -1\leqq y\leqq 0, -y-1\leqq x\leqq y+1, D_2: 0\leqq y\leqq 1, y-1\leqq x\leqq -y+1.$

与式 $=\displaystyle\int_{-1}^0\left\{\int_{-y-1}^{y+1}\sqrt{1-y^2}dx\right\}dy+\int_0^1\left\{\int_{y-1}^{-y+1}\sqrt{1-y^2}dx\right\}dy.$

$\displaystyle =\int_{-1}^0\sqrt{1-y^2}[x]_{-y-1}^{y+1}dy+\int_0^1\sqrt{1-y^2}[x]_{y-1}^{-y+1}dy$

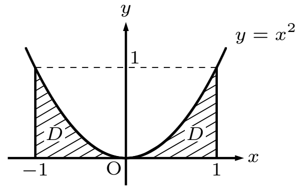
$\displaystyle =2\int_{-1}^0\sqrt{1-y^2}(y+1)dy+2\int_0^1\sqrt{1-y^2}(-y+1)dy.$ $1-y^2=t$ とおくと $-2ydy=dt\Rightarrow ydy=-\dfrac{1}{2}dt$. よって

$\displaystyle\int\sqrt{1-y^2}\,y\,dy=\int\sqrt{t}\left(-\frac{1}{2}dt\right)=-\frac{1}{2}\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}}=-\frac{1}{3}(1-y^2)^{\frac{3}{2}}$. 公式より $\displaystyle\int\sqrt{1-y^2}\,dy=\frac{1}{2}\left(y\sqrt{1-y^2}+\sin^{-1}y\right)$.

よって与式 $\displaystyle =2\left[-\frac{1}{3}(1-y^2)^{\frac{3}{2}}+\frac{1}{2}\left(y\sqrt{1-y^2}+\sin^{-1}y\right)\right]_{-1}^{0}+2\left[\frac{1}{3}(1-y^2)^{\frac{3}{2}}+\frac{1}{2}\left(y\sqrt{1-y^2}+\sin^{-1}y\right)\right]_{0}^{1}$

$\displaystyle =2\left(-\frac{1}{3}-\frac{1}{2}\sin^{-1}(-1)+\frac{1}{2}\sin^{-1}1-\frac{1}{3}\right)=\pi-\frac{4}{3}.$

128. $x^2\leqq 1\Rightarrow x^2-1\leqq 0\Rightarrow (x+1)(x-1)\leqq 0\Rightarrow -1\leqq x\leqq 1$. $D:-1\leqq x\leqq 1, 0\leqq y\leqq x^2$.




$x<0$ のとき $xe^{-y}<0$ より求める体積は

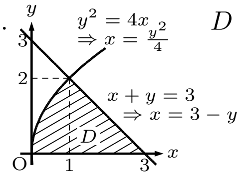
$\displaystyle\iint_D|xe^{-y}|dxdy=-\int_{-1}^{0}\left\{\int_0^{x^2}xe^{-y}dy\right\}dx+\int_0^1\left\{\int_0^{x^2}xe^{-y}dy\right\}dx$

$\displaystyle =-\int_{-1}^{0}x\left[-e^{-y}\right]_0^{x^2}dx+\int_0^1 x\left[-e^{-y}\right]_0^{x^2}dx=\int_{-1}^{0}\left(xe^{-x^2}-x\right)dx-\int_0^1\left(xe^{-x^2}-x\right)dx$. $-x^2=t$ とおくと

$-2xdx=dt\Rightarrow xdx=-\frac{1}{2}dt$. $\displaystyle\int xe^{-x^2}dx=\int e^t\left(-\frac{1}{2}dx\right)=-\frac{1}{2}e^t=-\frac{1}{2}e^{-x^2}$. よって求める体積は

$\displaystyle\left[-\frac{1}{2}e^{-x^2}-\frac{x^2}{2}\right]_{-1}^{0}-\left[-\frac{1}{2}e^{-x^2}-\frac{x^2}{2}\right]_0^1=-\frac{1}{2}-\left(-\frac{1}{2}e^{-1}-\frac{1}{2}\right)-\left(-\frac{1}{2}e^{-1}-\frac{1}{2}\right)+\left(-\frac{1}{2}\right)=e^{-1}=\frac{1}{e}.$

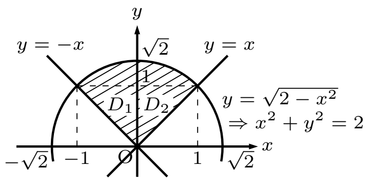
129.




$\displaystyle D:0\leqq y\leqq 2, \frac{y^2}{4}\leqq x\leqq 3-y$. $\displaystyle S=\iint_D dxdy=\int_0^2\left\{\int_{\frac{y^2}{4}}^{3-y}dx\right\}dy=\int_0^2[x]_{\frac{y^2}{4}}^{3-y}dy$

$\displaystyle =\int_0^2\left(3-y-\frac{y^2}{4}\right)dy=\left[3y-\frac{y^2}{2}-\frac{y^3}{12}\right]_0^2=6-2-\frac{2}{3}=\frac{10}{3}.$

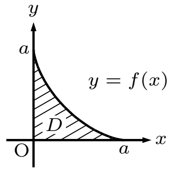
130. $D_1:-1\leqq x\leqq 0, -x\leqq y\leqq\sqrt{2-x^2}, D_2:0\leqq x\leqq 1, x\leqq y\leqq\sqrt{2-x^2}$.




与式 $\displaystyle =\iint_{D_1}dxdy+\iint_{D_2}dxdy$. 例題よりこれは $D_1$ と $D_2$ の面積だから合わせて

半径 $\sqrt{2}$ の円の面積の $\frac{1}{4}$. よって与式 $=\pi\sqrt{2}^2\times\frac{1}{4}=\frac{\pi}{2}$.

131.




$t=0$ のとき $(x,y)=(a,0)$, $t=\frac{\pi}{2}$ のとき $(x,y)=(0,a)$. よって曲線が $y=f(x)$ と表されるとすると $D:0\leqq x\leqq a, 0\leqq y\leqq f(x)$. $\displaystyle I=\int_0^a\left\{\int_0^{f(x)}xydy\right\}dx=\int_0^a\left[\frac{xy^2}{2}\right]_0^{f(x)}dx=\frac{1}{2}\int_0^a x\{f(x)\}^2dx.$

$x=a\cos^3t, f(x)=y=a\sin^3t$. よって $dx=3a\cos^2t(\cos t)'dt=-3a\sin t\cos^2t\,dt$. $\begin{array}{c|ccc} x & 0 & \to & a \\ \hline t & \frac{\pi}{2} & \to & 0 \end{array}$. よって

$\displaystyle I=\frac{1}{2}\int_{\frac{\pi}{2}}^{0}a\cos^3t\left\{a\sin^3t\right\}^2(-3a\sin t\cos^2t\,dt)=\frac{3a^4}{2}\int_0^{\frac{\pi}{2}}\sin^7t\cos^4t\cdot\cos t\,dt.$

$\sin t=u$ とおくと $\cos t\,dt=du$, $\cos^4t=(\cos^2t)^2=(1-\sin^2t)^2=(1-u^2)^2$, $\begin{array}{c|ccc} t & 0 & \to & \frac{\pi}{2} \\ \hline u & 0 & \to & 1 \end{array}$ よって

$\displaystyle I=\frac{3a^4}{2}\int_0^1u^7(1-u^2)^2du=\frac{3a^4}{2}\int_0^1(u^7-2u^9+u^{11})du=\frac{3a^4}{2}\left[\frac{u^8}{8}-\frac{u^{10}}{5}+\frac{u^{12}}{12}\right]_0^1$

$\displaystyle =\frac{3a^4}{2}\left(\frac{1}{8}-\frac{1}{5}+\frac{1}{12}\right)=\frac{a^4}{80}.$