p.32. 3 章§ 2. 変数変換と重積分　BASIC

132. 極座標による 2 重積分 $\iint_D f(x,y)dxdy = \iint_D f(r\cos\theta, r\sin\theta)rdrd\theta$

(1) $D: 0 \leqq r \leqq 3, 0 \leqq \theta \leqq \pi$ だから与式 $= \iint_D (r\sin\theta)rdrd\theta = \int_0^\pi \left\{\int_0^3 r^2 \sin\theta dr\right\} d\theta = \int_0^\pi \left[\frac{r^3}{3}\right]_0^3 \sin\theta d\theta$

$= \int_0^\pi 9\sin\theta d\theta = [-9\cos\theta]_0^\pi = -9\cos\pi + 9\cos 0 = 18.$

(2) $D: 0 \leqq r \leqq 1, 0 \leqq \theta \leqq 2\pi$ だから与式 $= \iint_D e^{r^2} rdrd\theta = \int_0^{2\pi} \left\{\int_0^1 re^{r^2} dr\right\} d\theta = \int_0^{2\pi} \left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta$

$= \int_0^{2\pi} \frac{1}{2}(e-1)d\theta = \frac{1}{2}(e-1)[\theta]_0^{2\pi} = -\pi(e-1).$

注：$(e^{r^2})' = e^{r^2}(r^2)' = 2re^{r^2}$ より $\int re^{r^2} = \frac{1}{2}e^{r^2} + C$. 正確には $r^2 = t$ とおいて置換積分.

133. 曲面 $z = 4a^2 - x^2 - y^2$ と xy 平面 $(z=0)$ の交線は $0 = 4a^2 - x^2 - y^2 \Rightarrow x^2 + y^2 = 4a^2$. よって極座標に変換して

$D: x^2 + y^2 \leqq 4a^2 \to 0 \leqq r \leqq 2a, 0 \leqq \theta \leqq 2\pi$ より $V = \iint_D (4a^2 - x^2 - y^2)dxdy = \iint_D (4a^2 - r^2)rdrd\theta$

$= \int_0^{2\pi} \left\{\int_0^{2a} (4a^2 r - r^3)dr\right\} d\theta = \int_0^{2\pi} \left[2a^2r^2 - \frac{r^4}{4}\right]_0^{2a} d\theta = \int_0^{2\pi} (8a^4 - 4a^4)d\theta = 4a^4[\theta]_0^{2\pi} = 8\pi a^4.$

134. 直円柱を曲面 $z = 4a^2 - x^2 - y^2$ と xy 平面 $(z=0)$ で切り取った部分の体積だから, 直円柱の底面 $D: (x-a)^2 + y^2 \leqq a^2$ 上で曲面 $z = 4a^2 - x^2 - y^2$ を積分すればよい. よって $V = \iint_D (4a^2 - x^2 - y^2)dxdy$.

極座標に変換すると $(x-a)^2 + y^2 = (r\cos\theta - a)^2 + r^2\sin^2\theta = r^2\cos^2\theta - 2ar\cos\theta + a^2 + r^2\sin^2\theta$

$= r^2(\cos^2\theta + \sin^2\theta) - 2ar\cos^2 + a^2 = r^2 - 2ar\cos\theta + a^2$ より

$D: r^2 - 2ar\cos\theta + a^2 \leqq a^2 \Rightarrow r^2 \leqq 2ra\cos\theta \Rightarrow r \leqq 2a\cos\theta$. $r \geqq 0$ より $0 \leqq r \leqq 2a\cos\theta$. この r が存在するのは

$0 \leqq \cos\theta$, すなわち $-\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}$ のときだから $D: -\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}, 0 \leqq r \leqq 2a\cos\theta$. よって

$V = \iint_D (4a^2 - r^2)rdrd\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{\int_0^{2a\cos\theta} (4a^2r - r^3)dr\right\} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[2a^2r^2 - \frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (8a^4\cos^2\theta - 4a^4\cos^4\theta)d\theta = 8a^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta d\theta - 4a^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta d\theta = 16a^4\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta - 8a^4\int_0^{\frac{\pi}{2}} \cos^4\theta d\theta$

$= 16a^4 \cdot \frac{1}{2} \cdot \frac{\pi}{2} - 8a^4 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5}{2}\pi a^4.$

注：教科書 p. 77 例題 2 も参照

135. $x + y = u \cdots$①$, 2x - y = v \cdots$②. ①+②, ①×2−② より $3x = u + v, 3y = 2u - v$. よって $x = \frac{u+v}{3}, y = \frac{2u-v}{3}$.

$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$. $D: 1 \leqq u \leqq 2, 1 \leqq v \leqq 3$. よって

与式 $= \iint_D \frac{v}{u}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \int_1^2 \left\{\int_1^3 \frac{v}{3u}dv\right\} du = \int_1^2 \left[\frac{v^2}{6u}\right]_1^3 du = \frac{1}{6}(9-1)\int_1^2 \frac{du}{u} = \frac{4}{3}[\log|u|]_1^2$

$= \frac{4}{3}(\log 2 - \log 1) = \frac{4}{3}\log 2.$

136. $\frac{1}{\sqrt{xy}}$ は $x = 0$ および $y = 0$ で定義されないので $D_{a,b} = \{(x,y) \mid a \leqq x \leqq 1, b \leqq y \leqq 1\}$ として

与式 $= \lim_{\substack{a\to+0 \\ b\to+0}} \iint_{D_{a,b}} \frac{1}{\sqrt{xy}}dxdy = \lim_{\substack{a\to+0 \\ b\to+0}} \int_a^1 \left\{\int_b^1 \frac{1}{\sqrt{xy}}dy\right\} dx = \lim_{\substack{a\to+0 \\ b\to}} \int_a^1 \frac{1}{\sqrt{x}} \left[\frac{y^{\frac{1}{2}}}{\frac{1}{2}}\right]_b^1 dx = \lim_{\substack{a\to+0 \\ b\to+0}} \int_a^1 \frac{1}{\sqrt{x}} 2(1-\sqrt{b})dx$

$= 2\lim_{a\to+0} \int_a^1 \frac{1}{\sqrt{x}}dx = 2\lim_{a\to+0} \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_a^1 = 2\lim_{a\to+0} 2(1-\sqrt{a}) = 4.$

137. 与式 $=\lim_{\substack{a\to\infty\\ b\to\infty}}\int_1^a\left\{\int_1^b\frac{1}{x^2y^2}dy\right\}dx=\lim_{\substack{a\to\infty\\ b\to\infty}}\int_1^a\frac{1}{x^2}\left[\frac{y^{-1}}{-1}\right]_1^b dx=\lim_{\substack{a\to\infty\\ b\to\infty}}\int_1^a\frac{1}{x^2}\left(-\frac{1}{b}+1\right)dx=\lim_{a\to\infty}\int_1^a\frac{1}{x^2}dx$

$=\lim_{a\to\infty}\left[\frac{x^{-1}}{-1}\right]_1^a=\lim_{a\to\infty}\left(-\frac{1}{a}+1\right)=1.$

138. $\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$

(1) $3x=t$ とおくと $3dx=dt\Rightarrow dx=\frac{1}{3}dt$, $\begin{array}{c|ccc} x & 0 & \to & \infty \\ \hline t & 0 & \to & \infty\end{array}$. 与式 $=\int_0^\infty e^{-t^2}\frac{1}{3}dt=\frac{1}{3}\int_0^\infty e^{-t^2}dt=\frac{\sqrt{\pi}}{6}.$

(2) 偶関数なので与式 $=2\int_0^\infty e^{-4x^2}dx$. $2x=t$ とおくと $2dx=dt\Rightarrow dx=\frac{1}{2}dt$, $\begin{array}{c|ccc} x & 0 & \to & \infty \\ \hline t & 0 & \to & \infty\end{array}$.

与式 $=2\int_0^\infty e^{-t^2}\frac{1}{2}dt=\int_0^\infty e^{-t^2}dt=\frac{\sqrt{\pi}}{2}.$

139. 曲面 $z=f(x,y)$ の領域 D に対応する部分の面積は $\iint_D\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}dxdy=\iint_D\sqrt{(z_x)^2+(z_y)^2+1}dxdy$

$z_x=1, z_y=2$. よって $S=\iint_D\sqrt{(z_x)^2+(z_y)^2+1}dxdy=\iint_D\sqrt{1^2+2^2+1}dxdy=\sqrt{6}\int_{-1}^1\left\{\int_{-1}^1 dy\right\}dx$

$=\sqrt{6}\int_{-1}^1[y]_{-1}^1dx=2\sqrt{6}\int_{-1}^1dx=2\sqrt{6}[x]_{-1}^1=4\sqrt{6}.$

140. $z_x=\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(x^2+y^2)_x=\frac{x}{\sqrt{x^2+y^2}}, z_y=\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(x^2+y^2)_y=\frac{y}{\sqrt{x^2+y^2}}$. よって

$\sqrt{(z_x)^2+(z_y)^2+1}=\sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2+1}=\sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1}=\sqrt{\frac{x^2+y^2}{x^2+y^2}+1}$

$=\sqrt{2}$. 極座標に変換すると $D:1\leqq r\leqq 2, 0\leqq\theta\leqq 2\pi$ だから $S=\iint_D\sqrt{(z_x)^2+(z_y)^2+1}dxdy=\iint_D\sqrt{2}dxdy$

$=\sqrt{2}\iint_D rdrd\theta=\sqrt{2}\int_0^{2\pi}\left\{\int_1^2 rdr\right\}d\theta=\sqrt{2}\int_0^{2\pi}\left[\frac{r^2}{2}\right]_1^2d\theta=\sqrt{2}\int_0^{2\pi}\left(2-\frac{1}{2}\right)d\theta=\frac{3}{2}\sqrt{2}[\theta]_0^{2\pi}=3\sqrt{2}\pi.$

141. 領域 D における $f(x,y)$ の平均は $\dfrac{\iint_D f(x,y)dxdy}{\iint_D dxdy}$

$D:0\leqq x\leqq 1, 0\leqq y\leqq 1-x$. $\iint_D(x+y)dxdy=\int_0^1\left\{\int_0^{1-x}(x+y)\right\}dx=\int_0^1\left[xy+\frac{y^2}{2}\right]_0^{1-x}dx$

$=\int_0^1\left\{x(1-x)+\frac{(1-x)^2}{2}\right)dx=\left[\frac{x^2}{2}-\frac{x^3}{3}-\frac{(1-x)^3}{6}\right]_0^1=\left(\frac{1}{2}-\frac{1}{3}-0\right)-\left(0-0-\frac{1}{6}\right)=\frac{1}{3}.$

$\iint_D dxdy=\int_0^1\left\{\int_0^{1-x}\right\}dx=\int_0^1[y]_0^{1-x}dx=\int_0^1(1-x))dx=\left[x-\frac{x^2}{2}\right]_0^1=1-\frac{1}{2}=\frac{1}{2}.$

よって求める平均は $\frac{\frac{1}{3}}{\frac{1}{2}}=\frac{2}{3}.$

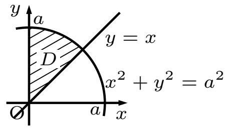
142. 図形 D の重心 $G(\overline{x},\overline{y})$ は $\overline{x}=\dfrac{\iint_D xdxdy}{\iint_D dxdy}, \overline{y}=\dfrac{\iint_D ydxdy}{\iint_D dxdy}$

$D:1\leqq x\leqq 2, 0\leqq y\leqq\frac{1}{x}$. $\iint_D xdxdy=\int_1^2\left\{\int_0^{\frac{1}{x}}xdy\right\}dx=\int_1^2[xy]_0^{\frac{1}{x}}dx=\int_1^2\left(x\cdot\frac{1}{x}\right)dx=\int_1^2dx=[x]_1^2=2-1=1.$

$\iint_D ydxdy=\int_1^2\left\{\int_0^{\frac{1}{x}}ydy\right\}dx=\int_1^2\left[\frac{y^2}{2}\right]_0^{\frac{1}{x}}dx=\int_1^2\frac{1}{2x^2}dx=\frac{1}{2}\left[\frac{x^{-1}}{-1}\right]_1^2=\frac{1}{2}\left(-\frac{1}{2}-1\right)=\frac{1}{4}.$

$\iint_D dxdy=\int_1^2\left\{\int_0^{\frac{1}{x}}dy\right\}dx=\int_1^2[y]_0^{\frac{1}{x}}dx=\int_1^2\frac{1}{x}dx=[\log|x|]_1^2=\log 2-\log 1=\log 2.$

よって $G\left(\frac{1}{\log 2},\frac{1}{4\log 2}\right).$

143.


極座標に変換すると $D: 0 \leqq r \leqq a, \dfrac{\pi}{4} \leqq \theta \leqq \dfrac{\pi}{2}$.

$$\iint_D x dxdy = \iint_D (r\cos\theta) r dr d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left\{ \int_0^a r^2 \cos\theta dr \right\} d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{r^3}{3} \right]_0^a \cos\theta d\theta = \frac{a^3}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\theta d\theta$$

$$= \frac{a^3}{3} [\sin\theta]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{a^3}{3} \left(\sin\frac{\pi}{2} - \sin\frac{\pi}{4} \right) = \frac{(2-\sqrt{2})a^3}{6}.$$

$$\iint_D y dxdy = \iint_D (r\sin\theta) r dr d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left\{ \int_0^a r^2 \sin\theta dr \right\} d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{r^3}{3} \right]_0^a \sin\theta d\theta = \frac{a^3}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\theta d\theta = \frac{a^3}{3} [-\cos\theta]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{a^3}{3} \left(-\cos\frac{\pi}{2} + \cos\frac{\pi}{4} \right) = \frac{\sqrt{2}a^3}{6}.$$

$$\iint_D dxdy = \iint_D r dr d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left\{ \int_0^a r dr \right\} d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{r^2}{2} \right]_0^a \sin\theta d\theta = \frac{a^2}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta = \frac{a^2}{2} [\theta]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi a^2}{8}.$$

$\overline{x} = \dfrac{\frac{(2-\sqrt{2})a^3}{6}}{\frac{\pi a^2}{8}} = \dfrac{4(2-\sqrt{2})a}{3\pi}, \overline{y} = \dfrac{\frac{\sqrt{2}a^3}{6}}{\frac{\pi a^2}{8}} = \dfrac{4\sqrt{2}a}{3\pi}$. よって重心は $\left(\dfrac{4(2-\sqrt{2})a}{3\pi}, \dfrac{4\sqrt{2}a}{3\pi} \right)$.

p.33 CHECK

144. $x = r\cos\theta, y = r\sin\theta$, $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$.

(1) $D: r^2 \leqq \pi^2$. $r \geqq 0, 0 \leqq \theta \leqq 2\pi$ より $D: 0 \leqq \theta \leqq 2\pi, 0 \leqq r \leqq \pi$. 部分積分を使って

$$与式 = \int_0^{2\pi} \left\{ \int_0^{\pi} \left(\sin\sqrt{r^2} \right) r dr \right\} d\theta = \int_0^{2\pi} \left\{ \int_0^{\pi} r \sin r dr \right\} d\theta = \int_0^{2\pi} \left\{ [r(-\cos r)]_0^{\pi} - \int_0^{\pi} (-\cos r) dr \right\} d\theta$$

$$= \int_0^{2\pi} \{ \pi(-\cos\pi) - [-\sin r]_0^{\pi} \} d\theta = (\pi + \sin\pi) \int_0^{2\pi} d\theta = \pi[\theta]_0^{2\pi} = 2\pi^2.$$

(2) $r^2 \leqq 1$. $y \geqq 0$ より $0 \leqq \theta \leqq \pi$. よって $D: 0 \leqq \theta \leqq \pi, 0 \leqq r \leqq 1$.

$$与式 = \int_0^{\pi} \left\{ \int_0^1 (r^2\cos^2\theta)(r\sin\theta) r dr \right\} d\theta = \int_0^{\pi} \left\{ \int_0^1 r^4 \cos^2\theta \sin\theta dr \right\} d\theta = \int_0^{\pi} \left[\frac{r^5}{5} \right]_0^1 \cos^2\theta \sin\theta d\theta$$

$= \dfrac{1}{5} \displaystyle\int_0^{\pi} \cos^2\theta \sin\theta d\theta$. $\cos\theta = t$ とおくと $-\sin\theta d\theta = dt \Rightarrow \sin\theta d\theta = -dt$.

θ	0	$\to$	π
t	1	$\to$	-1

$$与式 = \frac{1}{5} \int_1^{-1} t^2(-dt) = \frac{1}{5} \int_{-1}^1 t^2 dt = \frac{2}{5} \int_0^1 t^2 dt = \frac{2}{5} \left[\frac{t^3}{3} \right]_0^1 = \frac{2}{15}.$$

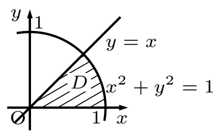
(3) $1 \leqq r^2 \leqq 9 \Rightarrow 1 \leqq r \leqq 3$. よって $D: 0 \leqq \theta \leqq 2\pi, 1 \leqq r \leqq 3$.

$$与式 = \int_0^{2\pi} \left\{ \int_1^3 \frac{1}{(r^2)^3} \cdot r dr \right\} d\theta = \int_0^{2\pi} \left\{ \int_1^3 r^{-5} dr \right\} d\theta = \int_0^{2\pi} \left[\frac{r^{-4}}{-4} \right]_1^3 d\theta = \int_0^{2\pi} \left(-\frac{1}{4 \cdot 3^4} + \frac{1}{4} \right) d\theta$$

$$= \frac{20}{81} [\theta]_0^{2\pi} = \frac{40}{81}\pi.$$

145. 問題の不等式より $r^2 \leqq 1, 0 \leqq r\sin\theta \leqq r\cos\theta$. よって

$r \leqq 1, 0 \leqq \sin\theta, \tan\theta \leqq 1 \Rightarrow D: 0 \leqq \theta \leqq \dfrac{\pi}{4}, 0 \leqq r \leqq 1$(直接グラフで確かめてもよい).



$$与式 = \int_0^{\frac{\pi}{4}} \left\{ \int_0^1 (r\sin\theta) r dr \right\} d\theta = \int_0^{\frac{\pi}{4}} \left\{ \int_0^1 r^2 \sin\theta dr \right\} d\theta = \int_0^{\frac{\pi}{4}} \left[\frac{r^3}{3} \right]_0^1 \sin\theta d\theta = \frac{1}{3} \int_0^{\frac{\pi}{4}} \sin\theta d\theta = \frac{1}{3} [-\cos\theta]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{3} \left(-\frac{1}{\sqrt{2}} + 1 \right) = \frac{1}{6}(2 - \sqrt{2}).$$

146. $D: x^2 + y^2 \leqq 4 \Rightarrow r^2 \leqq 4$. よって $D: 0 \leqq \theta \leqq 2\pi, 0 \leqq r \leqq 2$. D 上で $|x| \leqq 2$ より $4 - x^2 \geqq 0$ だから

$$与式 = \iint_D (4 - x^2) dxdy = \iint_D (4 - r^2\cos^2\theta) r dr d\theta = \int_0^{2\pi} \left\{ \int_0^2 (4r - r^3\cos^2\theta) dr \right\} d\theta = \int_0^{2\pi} \left[2r^2 - \frac{r^4}{4}\cos^2\theta \right]_0^2 d\theta$$

$$= \int_0^{2\pi} (8 - 4\cos^2\theta) d\theta = \int_0^{2\pi} \left(8 - 4 \cdot \frac{1 + \cos 2\theta}{2} \right) d\theta = \int_0^{2\pi} (6 - 2\cos 2\theta) d\theta = [6\theta - \sin 2\theta]_0^{2\pi} = 12\pi.$$

147. (1) $x + y = u, x - y = v$ とおくと $x = \dfrac{u+v}{2}, y = \dfrac{u-v}{2}$. よって $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\dfrac{1}{2}$.

$D: 0 \leqq u \leqq 2, 0 \leqq v \leqq 1$. 与式 $= \displaystyle\iint_D v^2 \left| -\frac{1}{2} \right| dudv = \frac{1}{2} \int_0^2 \left\{ \int_0^1 v^2 dv \right\} du = \frac{1}{2} \int_0^2 \left[\frac{v^3}{3} \right]_0^1 du = \frac{1}{6} [u]_0^2 = \frac{1}{3}$.

(2) $2x-y=u, x+y=v$ とおくと $x=\dfrac{u+v}{3}, y=\dfrac{2v-u}{3}$. よって $\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{vmatrix}=\dfrac{1}{3}$.

$D: 0 \leqq u \leqq 1, 1 \leqq v \leqq 2$. 与式 $=\displaystyle\iint_D \left(\frac{u+v}{3}-2\cdot\frac{2v-u}{3}\right)\left|\frac{1}{3}\right| dudv=\frac{1}{3}\int_0^1\left\{\int_1^2 (u-v)dv\right\}du$

$=\displaystyle\frac{1}{3}\int_0^1\left[uv-\frac{v^2}{2}\right]_1^2 du=\frac{1}{3}\int_0^1\left(u-\frac{3}{2}\right)du=\frac{1}{3}\left[\frac{u^2}{2}-\frac{3}{2}u\right]_0^1=-\frac{1}{3}$.

148. 極座標に変換すると $D: r \geqq 0, 0 \leqq \theta \leqq \dfrac{\pi}{2}$. よって与式 $=\displaystyle\lim_{a\to\infty}\int_0^{\frac{\pi}{2}}\left\{\int_0^a (1+r^2)^{-\frac{5}{2}} r dr\right\}d\theta$.

$1+r^2=t$ とおくと $2rdr=dt \Rightarrow rdr=\dfrac{1}{2}dt$, $\begin{array}{c|ccc} r & 0 & \to & a \\ \hline t & 1 & \to & 1+a^2 \end{array}$. $\displaystyle\int_0^a (1+r^2)^{-\frac{5}{2}} r dr=\int_1^{1+a^2} t^{-\frac{5}{2}}\frac{1}{2}dt$

$=\displaystyle\frac{1}{2}\left[\frac{t^{-\frac{3}{2}}}{-\frac{3}{2}}\right]_1^{1+a^2}=-\frac{1}{3}\left\{(1+a^2)^{-\frac{3}{2}}-1\right\}=\frac{1}{3}\left\{1-\frac{1}{(1+a^2)^{\frac{3}{2}}}\right\}\to\frac{1}{3}\ (a\to\infty)$. よって

与式 $=\displaystyle\frac{1}{3}\int_0^{\frac{\pi}{2}} d\theta=\frac{1}{3}[\theta]_0^{\frac{\pi}{2}}=\frac{\pi}{6}$.

149. (1) e^{-ax^2} は偶関数だから与式 $=2\displaystyle\int_0^\infty e^{-ax^2}dx$. $\sqrt{a}x=t$ とおくと $ax^2=t^2$. $\sqrt{a}dx=dt \Rightarrow dx=\dfrac{1}{\sqrt{a}}dt$.

$\begin{array}{c|ccc} x & 0 & \to & \infty \\ \hline t & 0 & \to & \infty \end{array}$ よって与式 $=2\displaystyle\int_0^\infty e^{-t^2}\frac{1}{\sqrt{a}}dt=\frac{2}{\sqrt{a}}\int_0^\infty e^{-t^2}dt=\sqrt{\frac{\pi}{a}}$.

(2) $x-1=t$ とおくと $dx=dt$. $\begin{array}{c|ccc} x & -\infty & \to & \infty \\ \hline t & -\infty & \to & \infty \end{array}$ よって与式 $=\displaystyle\int_{-\infty}^\infty e^{-t^2}dt=2\int_0^\infty e^{-t^2}dt=\sqrt{\pi}$ (e^{-t^2} は偶関数).

(3) $-x^2-4x=-(x^2+4x)=-\{(x+2)^2-4\}=-(x+2)^2+4$. よって $e^{-x^2-4x}=e^{-(x+2)^2+4}=e^4e^{-(x+2)^2}$.

$x+2=t$ とおくと (2) と同様にして与式 $=e^4\displaystyle\int_{-\infty}^\infty e^{-(x+2)^2}dx=e^4\int_{-\infty}^\infty e^{-t^2}dt=2e^4\int_0^\infty e^{-t^2}dt=\sqrt{\pi}e^4$.

150. $z_x=0, z_y=\dfrac{1}{2}(4-y^2)^{-\frac{1}{2}}(4-y^2)'=\dfrac{-2y}{2\sqrt{4-y^2}}=-\dfrac{y}{\sqrt{4-y^2}}$.

$(z_x)^2+(z_y)^2+1=\left(-\dfrac{y}{\sqrt{4-y^2}}\right)^2+1=\dfrac{y^2}{4-y^2}+1=\dfrac{y^2+(4-y^2)}{4-y^2}=\dfrac{4}{4-y^2}$.

$S=\displaystyle\iint_D\sqrt{(z_x)^2+(z_y)^2+1}dxdy=\iint_D\frac{2}{\sqrt{4-y^2}}dxdy=\int_0^4\left\{\int_0^1\frac{2}{\sqrt{4-y^2}}dy\right\}dx=\int_0^4 2\left[\sin^{-1}\frac{y}{2}\right]_0^1 dx$

$=2\left(\sin^{-1}\dfrac{1}{2}\right)[x]_0^4=2\cdot\dfrac{\pi}{\sqrt{6}}\cdot 4=\dfrac{4}{3}\pi$.

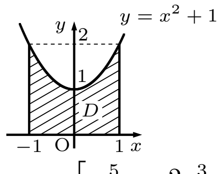
151. $z_x=y, z_y=x$. 極座標に変換すると $D: 0 \leqq \theta \leqq 2\pi, 0 \leqq r \leqq 1$.

$S=\displaystyle\iint_D\sqrt{(z_x)^2+(z_y)^2+1}dxdy=\iint_D\sqrt{y^2+x^2+1}dxdy=\iint_D\sqrt{r^2+1}rdrd\theta=\int_0^{2\pi}\left\{\int_0^1\sqrt{r^2+1}rdr\right\}d\theta$

$r^2+1=t$ とおくと $2rdr=dt \Rightarrow rdr=\dfrac{1}{2}dt$. $\begin{array}{c|ccc} r & 0 & \to & 1 \\ \hline t & 1 & \to & 2 \end{array}$ よって

$S=\displaystyle\int_0^{2\pi}\left\{\int_1^2\sqrt{t}\frac{1}{2}dt\right\}d\theta=\frac{1}{2}\int_0^{2\pi}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^2 d\theta=\frac{1}{3}\left(2^{\frac{3}{2}}-1\right)[\theta]_0^{2\pi}=\frac{2\pi}{3}(2\sqrt{2}-1)$.

152.



D は y 軸に関して対称だから $\overline{x}=0$.

$\displaystyle\iint_D ydxdy=\int_{-1}^1\left\{\int_0^{x^2+1}ydy\right\}dx=\int_{-1}^1\left[\frac{y^2}{2}\right]_0^{x^2+1}dx=\int_{-1}^1\frac{(x^2+1)^2}{2}dx=\int_0^1(x^4+2x^2+1)dx$

$=\left[\dfrac{x^5}{5}+\dfrac{2x^3}{3}+x\right]_0^1=\dfrac{1}{5}+\dfrac{2}{3}+1=\dfrac{28}{15}$.

$\displaystyle\iint_D dxdy=\int_{-1}^1\left\{\int_0^{x^2+1}dy\right\}dx=\int_{-1}^1[y]_0^{x^2+1}dx=\int_{-1}^1(x^2+1)dx=2\int_0^1(x^2+1)dx=2\left[\frac{x^3}{3}+x\right]=2\left(\frac{1}{3}+1\right)=\frac{8}{3}$.

よって $\overline{y}=\dfrac{\frac{28}{15}}{\frac{8}{3}}=\dfrac{7}{10}$. 従って求める重心 $\mathrm{G}(\overline{x},\overline{y})$ は $\left(0,\dfrac{7}{10}\right)$.