p.34. 3 章§ 2. 変数変換と重積分　STEPUP

153. (1) 極座標に変換すると $0 \leqq x \leqq y \Rightarrow 0 \leqq r\cos\theta \leqq r\sin\theta \Rightarrow 0 \leqq \cos\theta, 1 \leqq \tan\theta \Rightarrow \dfrac{\pi}{4} \leqq \theta \leqq \dfrac{\pi}{2}$

また $\dfrac{1}{2} \leqq x^2+y^2=r^2 \leqq 1 \Rightarrow \dfrac{1}{\sqrt{2}} \leqq r \leqq 1$. よって $D: \dfrac{\pi}{4} \leqq \theta \leqq \dfrac{\pi}{2}, \dfrac{1}{\sqrt{2}} \leqq r \leqq 1$.

与式 $= \displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left\{ \int_{\frac{1}{\sqrt{2}}}^{1} \frac{r\cos\theta}{r\sin\theta\sqrt{1+r^2}} rdr \right\} d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta} \left\{ \int_{\frac{1}{\sqrt{2}}}^{1} \frac{r}{\sqrt{1+r^2}} dr \right\} d\theta.$

$1+r^2=t$ とおくと $2rdr=dt \Rightarrow rdr = \dfrac{1}{2}dt$.

r	$\frac{1}{\sqrt{2}}$	$\to$	1
t	$\frac{3}{2}$	$\to$	2

$\displaystyle\int_{\frac{1}{\sqrt{2}}}^{1} \frac{r}{\sqrt{1+r^2}} dr = \int_{\frac{3}{2}}^{2} \frac{1}{\sqrt{t}} \cdot \frac{1}{2} dt = \frac{1}{2}\left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right]_{\frac{3}{2}}^{2} = \sqrt{2} - \sqrt{\frac{3}{2}}$. よって与式 $= \left(\sqrt{2}-\sqrt{\dfrac{3}{2}}\right)\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta} d\theta$

$\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta} d\theta = [\log|\sin\theta|]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \log\left|\sin\frac{\pi}{2}\right| - \log\left|\sin\frac{\pi}{4}\right| = \log 1 - \log\frac{1}{\sqrt{2}} = 0 - \log 2^{-\frac{1}{2}} = \frac{1}{2}\log 2.$

よって与式 $= \dfrac{1}{2}\left(\sqrt{2}-\sqrt{\dfrac{3}{2}}\right)\log 2 = \dfrac{1}{2}\left(\sqrt{2}-\dfrac{\sqrt{6}}{2}\right)\log 2 = \dfrac{1}{4}(2\sqrt{2}-\sqrt{6})\log 2.$

(2) 極座標に変換すると $x^2+(y-1)^2 = x^2+y^2-2y+1 \leqq 1 \Rightarrow r^2-2r\sin\theta+1 \leqq 1 \Rightarrow r^2 \leqq 2r\sin\theta \Rightarrow r \leqq 2\sin\theta$.

また $r \leqq 2\sin\theta$ より $\sin\theta \geqq 0$. $y \leqq x \Rightarrow r\sin\theta \leqq r\cos\theta \Rightarrow \tan\theta \leqq 1$ だから $0 \leqq \theta \leqq \dfrac{\pi}{4}$. よって

$D: 0 \leqq \theta \leqq \dfrac{\pi}{4}, 0 \leqq r \leqq 2\sin\theta$. 与式 $= \displaystyle\int_0^{\frac{\pi}{4}} \left\{ \int_0^{2\sin\theta} (4-r^2) rdr \right\} d\theta = \int_0^{\frac{\pi}{4}} \left[2r^2 - \frac{r^4}{4} \right]_0^{2\sin\theta} d\theta$

$= \displaystyle\int_0^{\frac{\pi}{4}} (8\sin^2\theta - 4\sin^4\theta) d\theta$. s'n$^2\theta = \dfrac{1-\cos 2\theta}{2}$ より $\sin^4\theta = \dfrac{(1-\cos 2\theta)^2}{4} = \dfrac{1-2\cos 2\theta+\cos^2 2\theta}{4}$

よって $8\sin^2\theta - 4\sin^4\theta = 4-4$ ɔs $2\theta - 1 + 2\cos 2\theta - \cos^2 2\theta = 3 - 2\cos 2\theta - \dfrac{1+\cos 4\theta}{2}$.

与式 $= \displaystyle\int_0^{\frac{\pi}{4}} \Big(\frac{5}{2} - 2\cos 2\theta -$ co ... $\Big) d\theta = \left[\frac{5}{2}\theta - \sin 2\theta - \frac{\sin 4\theta}{8} \right]_0^{\frac{\pi}{4}} = \frac{5}{8}\pi - \sin\frac{\pi}{2} - \frac{\sin\pi}{8} = \frac{5}{8}\pi - 1.$

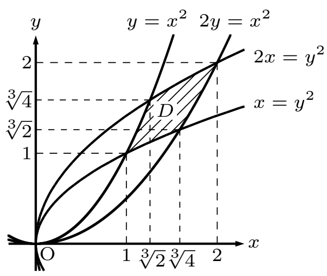
(3) 極座標に変換すると x^2+y^2 ... $r^2 \leqq 4 \Rightarrow r \leqq 2$. $x^2+y^2 \geqq 2x \Rightarrow r^2 \geqq 2r\cos\theta \Rightarrow r \geqq 2\cos\theta$.

$x \geqq 0 \Rightarrow r\cos\theta \geqq 0 \Rightarrow \cos$ ... $-\dfrac{\pi}{2} \leqq \theta \leqq \dfrac{\pi}{2}$. よって $D: -\dfrac{\pi}{2} \leqq \theta \leqq \dfrac{\pi}{2}, 2\cos\theta \leqq r \leqq 2$.

与式 $= \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Big\{ \int_{2\cos\theta}^{2} \sqrt{r^2} r d$ ... $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_{2\cos\theta}^{2} d\theta = \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (8-8\cos^3\theta) d\theta = \frac{16}{3}\int_0^{\frac{\pi}{2}} (1-\cos^3\theta) d\theta$

$= \dfrac{16}{3}\Big\{ [\theta]_0^{\frac{\pi}{2}} - \displaystyle\int_0^{\frac{\pi}{2}}$ (... $l\theta \Big\} = \dfrac{16}{3}\left(\dfrac{\pi}{2} - \dfrac{2}{3}\right) = \dfrac{8}{9}(3\pi-4).$

154. (1)



(2) $u^2v = x^3, uv^2 = y^3$ より $x = \sqrt[3]{u^2v} = u^{\frac{2}{3}}v^{\frac{1}{3}}, y = \sqrt[3]{uv^2} = u^{\frac{1}{3}}v^{\frac{2}{3}}$. よって $x_u = \dfrac{2}{3}u^{-\frac{1}{3}}v^{\frac{1}{3}}, x_v = \dfrac{1}{3}u^{\frac{2}{3}}v^{-\frac{2}{3}}$,

$y_u = \dfrac{1}{3}u^{-\frac{2}{3}}v^{\frac{2}{3}}, y_v = \dfrac{2}{3}u^{\frac{1}{3}}v^{-\frac{1}{3}}$. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{2}{3}u^{-\frac{1}{3}}v^{\frac{1}{3}} & \frac{1}{3}u^{\frac{2}{3}}v^{-\frac{2}{3}} \\ \frac{1}{3}u^{-\frac{2}{3}}v^{\frac{2}{3}} & \frac{2}{3}u^{\frac{1}{3}}v^{-\frac{1}{3}} \end{vmatrix} = \dfrac{4}{9} - \dfrac{1}{9} = \dfrac{1}{3}.$

$D: y \leqq x^2 \leqq 2y, x \leqq y^2 \leqq 2x \Rightarrow D: 1 \leqq \dfrac{x^2}{y} = u \leqq 2, 1 \leqq \dfrac{y^2}{x} = v \leqq 2$.　$xy = uv$ だから

与式 $= \displaystyle\int_1^2 \Big\{ \int_1^2 uv \left|\frac{1}{3}\right| dv \Big\} du = \frac{1}{3}\int_1^2 \left[\frac{uv^2}{2}\right]_1^2 du = \frac{1}{3}\int_1^2 \frac{3}{2}u du = \frac{1}{2}\left[\frac{u^2}{2}\right]_1^2 = \frac{3}{4}.$

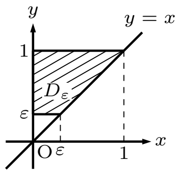
155. $x = 2u, y = 3v$ とおくと $D: u \geqq 0, v \geqq 0, u^2+v^2 \leqq 1$. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$. $xy = 6uv$. よって

与式 $=\displaystyle\iint_D 6uv|6|dudv$. ここで u,v を極座標に変換すると $r\cos\theta \geqq 0, r\sin\theta \geqq 0, r^2 \leqq 1$ より

$D: 0 \leqq \theta \leqq \dfrac{\pi}{2}, 0 \leqq r \leqq 1$. 与式 $= 36\displaystyle\int_0^{\frac{\pi}{2}}\left\{\int_0^1 r^2(\sin\theta\cos\theta)rdr\right\}d\theta = 18\int_0^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^1 \sin 2\theta d\theta = \frac{9}{2}\left[\frac{-\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}}$.

$=\dfrac{9}{4}\{-\cos\pi + \cos 0\} = \dfrac{9}{2}$.

156. $(0,0)$ のとき $\dfrac{1}{x^2+y^2}$ は定義できないので $D_\varepsilon: 0 \leqq x \leqq y \leqq 1, y \geqq \varepsilon$ とすると



与式 $=\displaystyle\lim_{\varepsilon\to+0}\iint_{D_\varepsilon}\frac{1}{\sqrt{x^2+y^2}}dxdy$. $D_\varepsilon: \varepsilon \leqq y \leqq 1, 0 \leqq x \leqq y$ より

与式 $=\displaystyle\lim_{\varepsilon\to+0}\int_\varepsilon^1\left\{\int_0^y \frac{1}{\sqrt{x^2+y^2}}dx\right\}dy$. 公式 $\displaystyle\int\frac{dx}{\sqrt{x^2+A}} = \log\left|x+\sqrt{x^2+A}\right|$ より

与式 $=\displaystyle\lim_{\varepsilon\to+0}\int_\varepsilon^1\left[\log\left|x+\sqrt{x^2+y^2}\right|\right]_0^y dy = \lim_{\varepsilon\to+0}\int_\varepsilon^1\left(\log\left|y+\sqrt{2y^2}\right| - \log\sqrt{y^2}\right)dy = \lim_{\varepsilon\to+0}\int_\varepsilon^1\log\left|\frac{y+y\sqrt{2}}{y}\right|dy$.

$=\log\left(1+\sqrt{2}\right)\displaystyle\lim_{\varepsilon\to+0}[y]_\varepsilon^1 = \log\left(1+\sqrt{2}\right)\lim_{\varepsilon\to+0}(1-\varepsilon) = \log\left(1+\sqrt{2}\right)$. (極座標に変換して求めることも可能)

157. $D: x^2+y^2 \leqq 4x$. $V=\displaystyle\iint_D xy^2dxdy$. 極座標に変換すると $r^2 \leqq 4r\cos\theta$ より $r \leqq 4\cos\theta$. よって

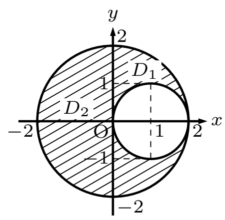
$\cos\theta \geqq 0$ より $-\dfrac{\pi}{2} \leqq \theta \leqq \dfrac{\pi}{2}$. $D: -\dfrac{\pi}{2} \leqq \theta \leqq \dfrac{\pi}{2}, 0 \leqq r \leqq 4\cos\theta$.

与式 $=\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\int_0^{4\cos\theta}(r^3\cos\theta\sin^2\theta)rdr\right\}d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{r^5}{5}\right]_0^{4\cos\theta}\cos\theta\sin^2\theta d\theta = \frac{1024}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^6\theta\sin^2\theta d\theta$

$=\dfrac{2048}{5}\displaystyle\int_0^{\frac{\pi}{2}}\cos^6\theta\sin^2\theta d\theta = \frac{2048}{5}\int_0^{\frac{\pi}{2}}\cos^6\theta(1-\cos^2\theta)d\theta = \frac{2048}{5}\left(\int_0^{\frac{\pi}{2}}\cos^6\theta d\theta - \int_0^{\frac{\pi}{2}}\cos^8\theta d\theta\right)$

$=\dfrac{2048}{5}\left(\dfrac{5}{6}\cdot\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\dfrac{\pi}{2} - \dfrac{7}{8}\cdot\dfrac{5}{6}\cdot\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\dfrac{\pi}{2}\right) = 8\pi$.

158. $x^2+y^2=2x \Rightarrow x^2-2x+y^2=0 \Rightarrow (x-1)^2+y^2=1$. $D: x^2+y^2 \leqq 4, x^2+y^2 \geqq 2x$.



D を $D=D_1+D_2$, $D_1: x \geqq 0, 2x \leqq x^2+y^2 \leqq 4$, $D_2: x \leqq 0, x^2+y^2 \leqq 4$ とする.

極座標に変換すると $D_2: \dfrac{\pi}{2} \leqq \theta \leqq \dfrac{3}{2}\pi, 0 \leqq r \leqq 2$. D_1 は $2r\cos\theta \leqq r^2 \leqq 4$ より

$D_1: -\dfrac{\pi}{2} \leqq \theta \leqq \dfrac{\pi}{2}, 2\cos\theta \leqq r \leqq 2$. D が x 軸に関して対称だから $\overline{y}=0$.

$\displaystyle\iint_D xdxdy = \iint_{D_1}xdxdy + \iint_{D_2}xdxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\int_{2\cos\theta}^2(r\cos\theta)rdr\right\}d\theta + \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\left\{\int_0^2(r\cos\theta)rdr\right\}d\theta$

$=\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{r^3}{3}\right]_{2\cos\theta}^2\cos\theta d\theta + \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\left[\frac{r^3}{3}\right]_0^2\cos\theta d\theta = \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\cos^3\theta)\cos\theta d\theta + \frac{8}{3}\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\cos\theta d\theta$

$=\dfrac{16}{3}\displaystyle\int_0^{\frac{\pi}{2}}(\cos\theta-\cos^4\theta)d\theta + \frac{8}{3}[\sin\theta]_{\frac{\pi}{2}}^{\frac{3}{2}\pi} = \frac{16}{3}\left([\sin\theta]_0^{\frac{\pi}{2}} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) + \frac{8}{3}\left(\sin\frac{3}{2}\pi - \sin\frac{\pi}{2}\right) = \frac{16}{3} - \pi - \frac{16}{3}$

$=-\pi$. $\displaystyle\iint_D dxdy = D$ の面積 $=4\pi-\pi=3\pi$ だから $\overline{x} = \dfrac{-\pi}{3\pi} = -\dfrac{1}{3}$. よって D の重心 G は $G\left(-\dfrac{1}{3}, 0\right)$.

159. (1) 与式 $=\displaystyle\int_{-\infty}^{\infty}(x^2-2x+1)e^{-x^2}dx = \int_{-\infty}^{\infty}x^2e^{-x^2}dx - 2\int_{-\infty}^{\infty}xe^{-x^2}dx + \int_{-\infty}^{\infty}e^{-x^2}dx$

xe^{-x^2} は奇関数, $x^2e^{-x^2}, e^{-x^2}$ は偶関数だから例題より

与式 $=2\displaystyle\int_0^{\infty}x^2e^{-x^2}dx + 2\int_0^{\infty}e^{-x^2}dx = 2\cdot\frac{\sqrt{\pi}}{4} + 2\cdot\frac{\sqrt{\pi}}{2} = \frac{3}{2}\sqrt{\pi}$.

(2) $\dfrac{x}{\sqrt{2}} = t$ とおくと $\dfrac{x^2}{2} = t^2, x=\sqrt{2}t, dx=\sqrt{2}dt$.

x	$0 \to \infty$
t	$0 \to \infty$

与式 $=\displaystyle\int_0^{\infty}(\sqrt{2}t)^2e^{-t^2}\cdot\sqrt{2}dt = 2\sqrt{2}\int_0^{\infty}t^2e^{-t^2}dt = 2\sqrt{2}\cdot\frac{\sqrt{\pi}}{4} = \frac{\sqrt{2\pi}}{2}\left(=\sqrt{\frac{\pi}{2}}\right)$.

(3) $\sqrt{x}=t$ とおくと $x=t^2, dx=2tdt$.

x	$0 \to \infty$
t	$0 \to \infty$

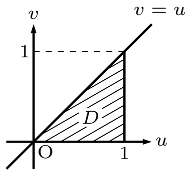
与式 $=\displaystyle\int_0^{\infty}te^{-t^2}2tdt = 2\int_0^{\infty}t^2e^{-t^2}dt = 2\cdot\frac{\sqrt{\pi}}{4} = \frac{\sqrt{\pi}}{2}$.

(4) $\log\dfrac{1}{x} = t$ とおくと $\log\dfrac{1}{x} = \log x^{-1} = -\log x = t, \log x = -t, x=e^{-t}, dx=(e^{-t})'dt = -e^{-t}dt$.

x	$0 \to 1$
t	$\infty \to 0$

与式 $=\displaystyle\int_{\infty}^0\sqrt{t}(-e^{-t}dt) = \int_0^{\infty}\sqrt{t}e^{-t}dt$. (3) より与式 $=\dfrac{\sqrt{\pi}}{2}$.

160. $x+y=u, y=v$ とおくと $x=u-y=u-v$. よって D は $u-v \geqq 0 \Rightarrow v \leqq u$ より

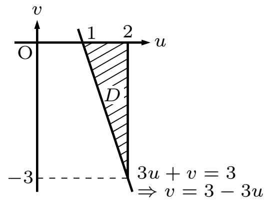


$D: v \geqq 0, v \leqq u \leqq 1$. 右図より $D: 0 \leqq u \leqq 1, 0 \leqq v \leqq u$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1.\ \text{与式} = \iint_D e^{-u^2} du dv = \int_0^1 \left\{ \int_0^u e^{-u^2} dv \right\} du$$

$$= \int_0^1 \left\{ \int_0^u e^{-u^2} du \right\} dv = \int_0^1 e^{-u^2} [v]_0^u du = \int_0^1 e^{-u^2} u du = \left[-\frac{1}{2} e^{-u^2} \right]_0^1 = -\frac{1}{2} e^{-1} + \frac{1}{2} = \frac{1}{2}\left(1 - \frac{1}{e}\right).$$

161. $y+x^2=u, y-x^2=v$ とすると $u+v=2y, u-v=2x^2 \Rightarrow y=\dfrac{u+v}{2}, x^2=\dfrac{u-v}{2} \cdots$①.



D は $y \leqq x^2 \Rightarrow y-x^2=v \leqq 0$, $\quad y \leqq -x^2+2 \Rightarrow y+x^2=u \leqq 2$,

$y \geqq -\dfrac{1}{2}x^2+\dfrac{3}{4} \Rightarrow \dfrac{u+v}{2} \geqq -\dfrac{u-v}{4}+\dfrac{3}{4} \Rightarrow 3u+v \geqq 3$ より

$D: v \leqq 0, u \leqq 2, 3u+v \geqq 3$. 右図より $D: 1 \leqq u \leqq 2, 3-3u \leqq v \leqq 0$.

$(x^2)_u = 2xx_u, (x^2)_v = 2xx_v$ だから①より $2xx_u = \dfrac{1}{2}, 2xx_v = -\dfrac{1}{2} \Rightarrow x_u = \dfrac{1}{4x}, x_v = -\dfrac{1}{4x}$.

また $y_u = \dfrac{1}{2}, y_v = \dfrac{1}{2}$ より $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{4x} & -\frac{1}{4x} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \dfrac{1}{8x} - \left(-\dfrac{1}{8x}\right) = \dfrac{1}{4x}$.

$dxdy = \left| \dfrac{\partial(x,y)}{\partial(u,v)} \right| dudv = \left| \dfrac{1}{4x} \right| dudv$. $x>0$ より $xdxdy = \dfrac{1}{4} dudv$. よって

$$\text{与式} = \iint_D \frac{v}{u^4} \frac{1}{4} dudv = \frac{1}{4} \int_1^2 \left\{ \int_{3-3u}^0 \frac{v}{u^4} dv \right\} du = \frac{1}{4} \int_1^2 \left[\frac{v^2}{2u^4} \right]_{3-3u}^0 du = \frac{1}{8} \int_1^2 \left\{ 0 - \frac{(3-3u)^2}{u^4} \right\} du$$

$$= -\frac{9}{8} \int_1^2 \left(\frac{1}{u^4} - \frac{2}{u^3} + \frac{1}{u^2} \right) du = -\frac{9}{8} \left[\frac{u^{-3}}{-3} - \frac{2u^{-2}}{-2} + \frac{u^{-1}}{-1} \right]_1^2 = \frac{9}{8} \left\{ \left(\frac{1}{24} - \frac{1}{4} + \frac{1}{2} \right) - \left(\frac{1}{3} - 1 + 1 \right) \right\}$$

$$= -\frac{3}{64}.$$

p.36 PLUS

1 座標軸の回転

162. (1) $x = X\cos\dfrac{\pi}{3} - Y\sin\dfrac{\pi}{3} = \dfrac{X-\sqrt{3}Y}{2}, y = X\sin\dfrac{\pi}{3} + Y\cos\dfrac{\pi}{3} = \dfrac{\sqrt{3}X+Y}{2}$ より

$$2\left(\frac{X-\sqrt{3}Y}{2}\right)^2 - \sqrt{3}\left(\frac{X-\sqrt{3}Y}{2}\right)\left(\frac{\sqrt{3}X+Y}{2}\right) + \left(\frac{\sqrt{3}X+Y}{2}\right)^2 = 1.$$

$$\frac{2(X-\sqrt{3}Y)^2 - \sqrt{3}(X-\sqrt{3}Y)(\sqrt{3}X+Y) + (\sqrt{3}X+Y)^2}{4} = 1.$$

$2X^2 - 4\sqrt{3}XY + 6Y^2 - 3X^2 + 2\sqrt{3}XY + 3Y^2 + 3X^2 + 2\sqrt{3}XY + Y^2 = 4$. $2X^2 + 10Y^2 = 4$. $X^2 + 5Y^2 = 2$.

(2) $x = X\cos\dfrac{\pi}{6} - Y\sin\dfrac{\pi}{6} = \dfrac{\sqrt{3}X-Y}{2}, y = X\sin\dfrac{\pi}{6} + Y\cos\dfrac{\pi}{6} = \dfrac{X+\sqrt{3}Y}{2}$ より

$$\left(\frac{\sqrt{3}X-Y}{2}\right)^2 + 2\sqrt{3}\left(\frac{\sqrt{3}X-Y}{2}\right)\left(\frac{X+\sqrt{3}Y}{2}\right) - \left(\frac{X+\sqrt{3}Y}{2}\right)^2 = 1.$$

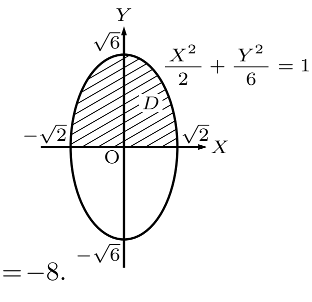
$$\frac{(\sqrt{3}X-Y)^2 + 2\sqrt{3}(\sqrt{3}X-Y)(X+\sqrt{3}Y) - (X+\sqrt{3}Y)^2}{4} = 1.$$

$3X^2 - 2\sqrt{3}XY + Y^2 + 6X^2 + 4\sqrt{3}XY - 6Y^2 - X^2 - 2\sqrt{3}XY - 3Y^2 = 4$. $8X^2 - 8Y^2 = 4$. $2X^2 - 2Y^2 = 1$.

163. $x = X\cos\dfrac{\pi}{4} - Y\sin\dfrac{\pi}{4} = \dfrac{X-Y}{\sqrt{2}}, y = X\sin\dfrac{\pi}{4} + Y\cos\dfrac{\pi}{4} = \dfrac{X+Y}{\sqrt{2}}$ より

$$x^2+xy+y^2 = \frac{(X-Y)^2 + (X-Y)(X+Y) + (X+Y)^2}{2} = \frac{X^2 - 2XY + Y^2 + X^2 - Y^2 + X^2 + 2XY + Y^2}{2}$$

$= \dfrac{3X^2+Y^2}{2} \leqq 3$. $y \geqq x \Rightarrow \dfrac{X+Y}{\sqrt{2}} \geqq \dfrac{X-Y}{\sqrt{2}} \Rightarrow Y \geqq 0$. $D: \dfrac{X^2}{2} + \dfrac{Y^2}{6} \leqq 1, Y \geqq 0$.



$x-y = \dfrac{X-Y}{\sqrt{2}} - \dfrac{X+Y}{\sqrt{2}} = -\sqrt{2}Y$, $D: -\sqrt{2} \leqq X \leqq \sqrt{2}, 0 \leqq Y \leqq \sqrt{6-3X^2}$ より

$$\text{与式} = -\iint_D \sqrt{2}Y dXdY = -\int_{-\sqrt{2}}^{\sqrt{2}} \left\{ \int_0^{\sqrt{6-3X^2}} \sqrt{2}Y dY \right\} dX = -\int_{-\sqrt{2}}^{\sqrt{2}} \left[\frac{\sqrt{2}Y^2}{2} \right]_0^{\sqrt{6-3X^2}} dX$$

$$= -\frac{\sqrt{2}}{2} \int_{-\sqrt{2}}^{\sqrt{2}} (6-3X^2) dX = -\sqrt{2} \int_0^{\sqrt{2}} (6-3X^2) dX = -\sqrt{2}[6X - X^3]_0^{\sqrt{2}} = -\sqrt{2}(6\sqrt{2} - 2\sqrt{2}) = -8.$$

2 3 重積分

164. (1) 与式 $=\displaystyle\int_0^1\left\{\int_0^x\left\{\int_0^{xy}x^2y^2z dz\right\}dx\right\}dy=\int_0^1\left\{\int_0^x x^2y^2\left[\frac{z^2}{2}\right]_0^{xy}dx\right\}dy=\frac{1}{2}\int_0^1\left\{\int_0^x x^4y^4dy\right\}dx$

$\displaystyle =\frac{1}{2}\int_0^1 x^4\left[\frac{y^5}{5}\right]_0^x dx=\frac{1}{10}\int_0^1 x^9dx=\frac{1}{10}\left[\frac{x^{10}}{10}\right]_0^1=\frac{1}{100}.$

(2) $D: x^2+y^2\leqq 1,\ V: D, -\sqrt{1-x^2-y^2}\leqq z\leqq\sqrt{1-x^2-y^2}.$

与式 $=\displaystyle\iint_D\left\{\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}z^2dz\right\}dxdy=2\iint_D\left[\frac{z^3}{3}\right]_0^{\sqrt{1-x^2-y^2}}dxdy=\frac{2}{3}\iint_D\sqrt{1-x^2-y^2}^3dxdy$

極座標に変換すると $D: 0\leqq\theta\leqq 2\pi, 0\leqq r\leqq 1.$ 与式 $=\displaystyle\frac{2}{3}\int_0^{2\pi}\left\{\int_0^1(1-r^2)^{\frac{3}{2}}rdr\right\}d\theta.$ $1-r^2=t$ とおくと

$-2rdr=dt\Rightarrow rdr=-\dfrac{1}{2}dt.$

r	$0\ \to\ 1$
t	$1\ \to\ 0$

より与式 $=\displaystyle\frac{2}{3}\int_0^{2\pi}\left\{\int_1^0 t^{\frac{3}{2}}\left(-\frac{1}{2}dt\right)\right\}d\theta=\frac{1}{3}\int_0^{2\pi}\left[\frac{t^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1 d\theta$

$\displaystyle =\frac{2}{15}[\theta]_0^{2\pi}=\frac{4}{15}\pi.$

3 3 重積分の変数変換

165. 球面座標に変換すると $x^2+y^2+z^2=r^2\sin^2\theta\cos^2\varphi+r^2\sin^2\theta\sin^2\varphi+r^2\cos^2\theta$

$=r^2\sin^2\theta(\cos^2\varphi+\sin^2\varphi)+r^2\cos^2\theta=r^2\sin^2\theta+r^2\cos^2\theta=r^2(\sin^2\theta+\cos^2\theta)=r^2.$

V は $r^2\leqq 1$ より $V: 0\leqq\theta\leqq\pi, 0\leqq\varphi\leqq 2\pi, 0\leqq r\leqq 1.$

与式 $=\displaystyle\int_0^{\pi}\left\{\int_0^{2\pi}\left\{\int_0^1 r^2\cdot r^2\sin\theta dr\right\}d\varphi\right\}d\theta=\int_0^{\pi}\left\{\int_0^{2\pi}\left[\frac{r^5}{5}\right]_0^1\sin\theta d\varphi\right\}d\theta=\frac{1}{5}\int_0^{\pi}[\varphi]_0^{2\pi}\sin\theta d\theta=\frac{2}{5}\pi[-\cos\theta]_0^{\pi}$

$\displaystyle =\frac{2}{5}\pi(-\cos\pi+\cos 0)=\frac{4}{5}\pi.$

166. 球面座標に変換すると $x^2+y^2+z^2=r^2, x=r\sin\theta\cos\varphi, y=r\sin\theta\sin\varphi, z=r\cos\theta$ より

$r^2\leqq R^2, r\sin\theta\cos\varphi\geqq 0, r\sin\theta\sin\varphi\geqq 0, r\cos\theta\geqq 0.$ $r\geqq 0$ より $0\leqq r\leqq R.$ また

$\sin\theta\cos\varphi\geqq 0, \sin\theta\sin\varphi\geqq 0, \cos\theta\geqq 0.$ $0\leqq\theta\leqq\pi$ より $\sin\theta\geqq 0.$ よって $\cos\varphi\geqq 0, \sin\varphi\geqq 0.$ 以上により

$V: 0\leqq\theta\leqq\dfrac{\pi}{2}, 0\leqq\varphi\leqq\dfrac{\pi}{2}, 0\leqq r\leqq R.$ 与式 $=\displaystyle\int_0^{\frac{\pi}{2}}\left\{\int_0^{\frac{\pi}{2}}\left\{\int_0^R(r^2\sin^2\theta\sin\varphi\cos\varphi)r^2\sin\theta dr\right\}d\varphi\right\}d\theta$

$\displaystyle =\int_0^{\frac{\pi}{2}}\left\{\int_0^{\frac{\pi}{2}}\left[\frac{r^5}{5}\right]_0^R\sin^3\theta\sin\varphi\cos\varphi d\varphi\right\}d\theta=\frac{R^5}{5}\int_0^{\frac{\pi}{2}}\sin^3\theta\left[\frac{\sin^2\varphi}{2}\right]_0^{\frac{\pi}{2}}d\theta=\frac{R^5}{10}\int_0^{\frac{\pi}{2}}\sin^3\theta d\theta=\frac{R^5}{10}\cdot\frac{2}{3}=\frac{R^5}{15}.$

167. (1) $J=\begin{vmatrix}x_r & x_\theta & x_z\\ y_r & y_\theta & y_z\\ z_r & z_\theta & z_z\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}=r\cos^2\theta-(-r\sin^2\theta)=r(\cos^2\theta+\sin^2\theta)=r.$

(2) $x^2+y^2=r^2\leqq 1$ より $V: 0\leqq\theta\leqq 2\pi, 0\leqq r\leqq 1, 0\leqq z\leqq 1.$

与式 $=\displaystyle\iiint_V(r^2+2z^2)rdzdrd\theta=\int_0^{2\pi}\left\{\int_0^1\left\{\int_0^1 r(r^2+2z^2)dz\right\}dr\right\}d\theta=\int_0^{2\pi}\left\{\int_0^1\left[r^3z+\frac{2rz^3}{3}\right]_0^1dr\right\}d\theta$

$\displaystyle =\int_0^{2\pi}\left\{\int_0^1\left(r^3+\frac{2}{3}r\right)dr\right\}d\theta=\int_0^{2\pi}\left[\frac{r^4}{4}+\frac{1}{3}r^2\right]_0^1 d\theta=\left(\frac{1}{4}+\frac{1}{3}\right)\int_0^{2\pi}d\theta=\frac{7}{12}[\theta]_0^{2\pi}=\frac{7}{6}\pi.$

4 ガンマ関数とベータ関数

168. (1) 例題 (2) より $\Gamma(4)=3!=6.$

(2) 例題 (1) より $\Gamma\left(\dfrac{7}{2}\right)=\dfrac{5}{2}\Gamma\left(\dfrac{5}{2}\right)=\dfrac{5}{2}\cdot\dfrac{3}{2}\Gamma\left(\dfrac{3}{2}\right)=\dfrac{5}{2}\cdot\dfrac{3}{2}\cdot\dfrac{1}{2}\Gamma\left(\dfrac{1}{2}\right).$ 例題 (3) より

$\Gamma\left(\dfrac{7}{2}\right)=\dfrac{5}{2}\cdot\dfrac{3}{2}\cdot\dfrac{1}{2}\sqrt{\pi}=\dfrac{15\sqrt{\pi}}{8}.$

(3) (2) より $\Gamma\left(\dfrac{9}{2}\right)=\dfrac{7}{2}\Gamma\left(\dfrac{7}{2}\right)=\dfrac{7}{2}\cdot\dfrac{15}{8}\sqrt{\pi}=\dfrac{105\sqrt{\pi}}{16}.$

169. $\dfrac{1}{1+x} = t$ とおくと $1+x = \dfrac{1}{t} \Rightarrow x = \dfrac{1}{t} - 1$ より $dx = (t^{-1})'dt = -t^{-2}dt = -\dfrac{dt}{t^2}$.

また $x = \dfrac{1}{t} - 1 = \dfrac{1-t}{t}$.

x	0	$\to$	∞
t	1	$\to$	0

与式 $= \displaystyle\int_1^0 \frac{\sqrt{\frac{1-t}{t}}}{(\frac{1}{t})^2} \left(-\frac{dt}{t^2}\right) = \int_0^1 \sqrt{\frac{1-t}{t}}dt = \int_0^1 t^{-\frac{1}{2}}(1-t)^{\frac{1}{2}}dt = B\left(\frac{1}{2}, \frac{3}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(2)} = \frac{\sqrt{\pi}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{1!} = \frac{\pi}{2}$.