

第3章 § 1 不定積分と定積分

p.95 練習問題 1-A

1. (1) 与式 $= \int \left(x - 2 + \frac{1}{x} - \frac{3}{x^2} \right) dx = \frac{1}{2}x^2 - 2x + \log|x| + \frac{3}{x} + C$

(2) 与式 $= \int \left(4x - 4 + \frac{1}{x} \right) dx = 2x^2 - 4x + \log|x| + C$

(3) 与式 $= \frac{1}{5}e^{5x} - \frac{1}{3}\cos 3x + C$

(4) 与式 $= \frac{1}{4}\log|4x+5| + C$

(5) 与式 $= \frac{1}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}} + C$

(6) 与式 $= \int \left(\frac{1}{x} + \frac{1}{\sqrt{x^2+1}} \right) dx = \log|x| + \log|x + \sqrt{x^2+1}| + C = \log|x(x + \sqrt{x^2+1})| + C$

2. (1) 与式 $= \int_0^2 (3x^3 - 6x^2) dx = \left[\frac{3}{4}x^4 - 2x^3 \right]_0^2 = 12 - 16 = -4$

(2) 与式 $= \int_1^4 (x^{\frac{3}{2}} - 3x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}) dx = \left[\frac{2}{5}x^{\frac{5}{2}} - 3 \cdot \frac{2}{3}x^{\frac{3}{2}} + 4 \cdot 2x^{\frac{1}{2}} \right]_1^4 = \frac{64}{5} - 16 + 16 - \left(\frac{2}{5} - 2 + 8 \right) = \frac{32}{5}$

(3) 与式 $= [e^x + \sin x]_0^{\frac{\pi}{2}} = e^{\frac{\pi}{2}} + \sin \frac{\pi}{2} - e^0 - \sin 0 = e^{\frac{\pi}{2}}$

(4) 与式 $= 2 \int_0^1 (5x^4 + x^2 + 1) dx = 2 \left[x^5 + \frac{1}{3}x^3 + x \right]_0^1 = 2 \left(1 + \frac{1}{3} + 1 \right) = \frac{14}{3}$

(5) 与式 $= 2 \int_0^1 \frac{1}{\sqrt{2-x^2}} dx = 2 \left[\sin^{-1} \frac{x}{\sqrt{2}} \right]_0^1 = 2 \sin^{-1} \frac{1}{\sqrt{2}} = \frac{\pi}{2}$

(6) 与式 $= \frac{1}{3} \int_0^1 \frac{1}{x^2 + \frac{1}{3}} dx = \frac{\sqrt{3}}{3} [\tan^{-1} \sqrt{3}x]_0^1 = \frac{1}{\sqrt{3}} \tan^{-1} \sqrt{3} = \frac{\pi}{3\sqrt{3}}$

3. $\int_{-1}^1 f(x) dx = \int_{-1}^1 (ax^2 + bx + c) dx = 2 \int_0^1 (ax^2 + c) dx = 2 \left[\frac{a}{3}x^3 + cx \right]_0^1 = \frac{2}{3}a + 2c = 1 \dots \textcircled{1}$

$\int_{-1}^1 xf(x) dx = \int_{-1}^1 (ax^3 + bx^2 + cx) dx = 2 \int_0^1 bx^2 dx = 2 \left[\frac{b}{3}x^3 \right]_0^1 = \frac{2}{3}b = 0 \dots \textcircled{2}$

$\int_{-1}^1 x^2 f(x) dx = \int_{-1}^1 (ax^4 + bx^3 + cx^2) dx = 2 \int_0^1 (ax^4 + cx^2) dx = 2 \left[\frac{a}{5}x^5 + \frac{c}{3}x^3 \right]_0^1 = \frac{2}{5}a + \frac{2}{3}c = 1 \dots \textcircled{3}$

$\textcircled{2}$ より $b = 0$. $\textcircled{1}$, $\textcircled{3}$ より $a = \frac{15}{4}$, $c = -\frac{3}{4}$

4. $\int \sinh x dx = \int \frac{e^x - e^{-x}}{2} dx = \frac{e^x - e^{-x}(-1)}{2} + C = \frac{e^x + e^{-x}}{2} + C = \cosh x + C$

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5. $y = x^3 - x$ と x 軸との交点の x 座標は $x^3 - x = x(x-1)(x+1) = 0$ より $x = 0, \pm 1$.

$-1 < x < 0$ のとき $x^3 - x > 0$. $0 < x < 1$ のとき $x^3 - x < 0$ よって求める面積 S は

$S = \int_{-1}^0 (x^3 - x) dx - \int_0^1 (x^3 - x) dx = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2 \right]_{-1}^0 - \left[\frac{1}{4}x^4 - \frac{1}{2}x^2 \right]_0^1 = -\left(\frac{1}{4} - \frac{1}{2} \right) - \left(\frac{1}{4} - \frac{1}{2} \right) = \frac{1}{2}$

p.96 練習問題 1-B

1. 左辺 $= \int_{\alpha}^{\beta} \{x^2 - (\alpha + \beta)x + \alpha\beta\} dx = \left[\frac{1}{3}x^3 - \frac{1}{2}(\alpha + \beta)x^2 + \alpha\beta x \right]_{\alpha}^{\beta}$
 $= \frac{1}{3}(\beta^3 - \alpha^3) - \frac{1}{2}(\alpha + \beta)(\beta^2 - \alpha^2) + \alpha\beta(\beta - \alpha) = \frac{1}{6}(\beta - \alpha)\{2\beta^2 + 2\alpha\beta + 2\alpha^2 - 3(\alpha + \beta)^2 + 6\alpha\beta\}$
 $= \frac{1}{6}(\beta - \alpha)(-\beta^2 + 2\alpha\beta - \alpha^2) = -\frac{1}{6}(\beta - \alpha)^3 = \text{右辺}$

2. $C = \int_{-1}^1 f(t) dt \dots \textcircled{1}$ とおくと $f(x) = 3x^2 - x + C$ よって $\int_{-1}^1 f(t) dt = \int_{-1}^1 (3t^2 - t + C) dt = 2 \int_0^1 (3t^2 + C) dt$
 $= 2[t^3 + Ct]_0^1 = 2 + 2C$. $\textcircled{1}$ より $C = 2 + 2C$ よって $C = -2$. ゆえに $f(x) = 3x^2 - x - 2$

3. $F(x)$ を $f(x)$ の不定積分の1つとすると $\frac{d}{dx} \int_x^{x+1} f(t) dt = \frac{d}{dx} [F(t)]_x^{x+1} = \frac{d}{dx} \{F(x+1) - F(x)\}$
 $= F'(x+1) - F'(x) = f(x+1) - f(x)$. $f(x) = ax^2 + bx + c$ とおくと

$$f(x+1) - f(x) = a(x+1)^2 + b(x+1) + c - ax^2 - bx - c = 2ax + a + b \text{ よって条件より } 2ax + a + b = 8x - 3$$

$$2a = 8, a + b = -3. \text{ よって } a = 4, b = -7, f(2) = 0 \text{ より } 4a + 2b + c = 0 \text{ よって } 16 - 14 + c = 0, c = -2$$

$$f(x) = 4x^2 - 7x - 2.$$

$$4. (1) \int_{-x}^x f(t) dt = \int_{-x}^0 f(t) dt + \int_0^x f(t) dt = -\int_0^{-x} f(t) dt + \int_0^x f(t) dt = -S(-x) + S(x) = S(x) - S(-x)$$

$$(2) (1) \text{ より } \frac{d}{dx} \int_{-x}^x f(t) dt = \frac{d}{dx} \{S(x) - S(-x)\} = S'(x) - S'(-x)(-x)' = f(x) + f(-x)$$

$$5. (1) 0 \leq x \leq 1 \text{ のとき } x^2 \leq x^{\frac{1}{2}} \leq x^0. \text{ つまり } x^2 \leq \sqrt{x} \leq 1 \text{ よって } 1 + x^2 \leq 1 + \sqrt{x} \leq 2, \frac{1}{1+x^2} \geq \frac{1}{1+\sqrt{x}} \geq \frac{1}{2}$$

$$(2) (1) \text{ で等号は常に成り立つことはないので } \int_0^1 \frac{1}{2} dx < \int_0^1 \frac{1}{1+\sqrt{x}} dx < \int_0^1 \frac{1}{1+x^2} dx$$

$$\int_0^1 \frac{1}{2} dx = \frac{1}{2} [x]_0^1 = \frac{1}{2}, \int_0^1 \frac{1}{1+x^2} dx = [\tan^{-1} x]_0^1 = \tan^{-1} 1 = \frac{\pi}{4}. \text{ よって } \frac{1}{2} < \int_0^1 \frac{1}{1+\sqrt{x}} dx < \frac{\pi}{4}$$

$$6. (1) OP = a, OQ = t \text{ より } PQ^2 + t^2 = a^2 \text{ よって } PQ = \sqrt{a^2 - t^2}. \text{ よって } \triangle OQP = \frac{1}{2} t \sqrt{a^2 - t^2}$$

$$\angle OPQ = \theta \text{ とすると } \angle BOP = \theta \text{ で扇形 OPB} = \frac{1}{2} a^2 \theta. \sin \theta = \frac{t}{a} \text{ だから } \theta = \sin^{-1} \frac{t}{a}$$

$$\text{よって扇形 OPB} = \frac{1}{2} a^2 \sin^{-1} \frac{t}{a}$$

(2) 定積分の値は $\triangle OQP$ と扇形 OPB の面積の和なので

$$\int_0^t \sqrt{a^2 - x^2} dx = \frac{1}{2} t \sqrt{a^2 - t^2} + \frac{1}{2} a^2 \sin^{-1} \frac{t}{a} = \frac{1}{2} \left(t \sqrt{a^2 - t^2} + a^2 \sin^{-1} \frac{t}{a} \right)$$