大日本図書　新微分積分 II　 2 章§ 1 偏微分法

p.39. 練習問題 1-A

1. $f_x(0,0)=\lim_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x-0}$. $x\to 0$ のとき $x\neq 0$ だから $(x,0)\neq(0,0)$. よって f の定義から
$f(x,0)=\dfrac{x^3-0^3}{x^2+0^2}=x, f(0,0)=0$. 従って $f_x(0,0)=\lim_{x\to 0}\dfrac{x-0}{x-0}=1$.
$f_y(0,0)=\lim_{y\to 0}\dfrac{f(0,y)-f(0,0)}{y-0}$. $y\to 0$ のとき $y\neq 0$ だから $(0,y)\neq(0,0)$. よって f の定義から
$f(0,y)=\dfrac{0^3-y^3}{0^2+y^2}=-y, f(0,0)=0$. 従って $f_y(0,0)=\lim_{y\to 0}\dfrac{-y-0}{y-0}=-1$.

2. (1) $z_x=\dfrac{(x^2y)_x(x-3y)-x^2y(x-3y)_x}{(x-3y)^2}=\dfrac{2xy(x-3y)-x^2y\cdot 1}{(x-3y)^2}=\dfrac{2x^2y-6xy^2-x^2y}{(x-3y)^2}=\dfrac{x^2y-6xy^2}{(x-3y)^2}$
$=\dfrac{xy(x-6y)}{(x-3y)^2}$.
$z_x=\dfrac{(x^2y)_y(x-3y)-x^2y(x-3y)_y}{(x-3y)^2}=\dfrac{x^2(x-3y)-x^2y\cdot(-3)}{(x-3y)^2}=\dfrac{x^3-3x^2y+3x^2y}{(x-3y)^2}=\dfrac{x^3}{(x-3y)^2}$

(2) $z_x=(x)'e^{-xy}+x(e^{-xy})_x=e^{-xy}+xe^{-xy}(-xy)_x=e^{-xy}+xe^{-xy}(-y)=e^{-xy}-xye^{-xy}=(1-xy)e^{-xy}$.
$z_y=x(e^{-xy})_y=xe^{-xy}(-xy)_y=xe^{-xy}(-x)=-x^2e^{-xy}$.

(3) $z_x=\dfrac{\{\cos(x-2y)\}_x}{\cos(x-2y}=\dfrac{\{-\sin(x-2y)\}(x-2y)_x}{\cos(x-2y}=\dfrac{-\sin(x-2y)}{\cos(x-2y}=-\tan(x-2y)$.
$z_y=\dfrac{\{\cos(x-2y)\}_y}{\cos(x-2y}=\dfrac{\{-\sin(x-2y)\}(x-2y)_y}{\cos(x-2y}=\dfrac{2\sin(x-2y)}{\cos(x-2y}=2\tan(x-2y)$.

(4) $z_x=2\sin(x+y)\{\sin(x+y)\}_x-2\sin x(\sin x)'=2\sin(x+y)\{\cos(x+y)\}(x+y)_x-2\sin x\cos x$
$=2\sin(x+y)\cos(x+y)-2\sin x\cos x=\sin 2(x+y)-\sin 2x$.
$z_y=2\sin(x+y)\{\sin(x+y)\}_y-2\sin y(\sin y)'=2\sin(x+y)\{\cos(x+y)\}(x+y)_y-2\sin y\cos y$
$=2\sin(x+y)\cos(x+y)-2\sin y\cos y=\sin 2(x+y)-\sin 2y$.

3. (1) $z_x=y(x^{-1})'-\dfrac{1}{y}=-yx^{-2}-\dfrac{1}{y}=-\dfrac{y}{x^2}-\dfrac{1}{y}=-\dfrac{y^2+x^2}{x^2y}, z_y=\dfrac{1}{x}-x(y^{-1})'=\dfrac{1}{x}+xy^{-2}=\dfrac{1}{x}+\dfrac{x}{y^2}$
$=\dfrac{y^2+x^2}{xy^2}$. よって $dz=-\dfrac{x^2+y^2}{x^2y}dx+\dfrac{x^2+y^2}{xy^2}dy$.

(2) $z_x=(x)'y\sin(x-y)+xy\{\sin(x-y)\}_x=y\sin(x-y)+xy\{\cos(x-y)\}(x-y)_x=y\sin(x-y)+xy\cos(x-y)$,
$z_y=x(y)'\sin(x-y)+xy\{\sin(x-y)\}_y=x\sin(x-y)+xy\{\cos(x-y)\}(x-y)_y=x\sin(x-y)-xy\cos(x-y)$.
よって $dz=\{y\sin(x-y)+xy\cos(x-y)\}dx+\{x\sin(x-y)-xy\cos(x-y)\}dy$.

4. (1) $z_x=8x, z_y=18y$. $(x,y)=(-2,-1)$ のとき $z_x=-16, z_y=-18$. よって求める接平面の方程式は
$z-25=-16(x+2)-18(y+1)$. よって $16x+18y+z=-25$.

(2) $z_x=\dfrac{1}{2}(3-x^2-y^2)^{-\frac{1}{2}}(3-x^2-y^2)_x=\dfrac{-2x}{2\sqrt{3-x^2-y^2}}=-\dfrac{x}{\sqrt{3-x^2-y^2}}, z_y=\dfrac{1}{2}(3-x^2-y^2)^{-\frac{1}{2}}(3-x^2-y^2)_y$
$=\dfrac{-2y}{2\sqrt{3-x^2-y^2}}=-\dfrac{y}{\sqrt{3-x^2-y^2}}$. $(x,y)=(1,1)$ のとき $z_x=-1, z_y=-1$. よって求める接平面の方程式は
$z-1=-(x-1)-(y-1)$. よって $x+y+z=3$.

(3) $z_x=\{\cos(x+y)\}(x+y)_x=\cos(x+y), z_y=\{\cos(x+y)\}(x+y)_y=\cos(x+y)$
$(x,y)=\left(\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$ のとき $z=\sin\left(\dfrac{\pi}{2}+\dfrac{\pi}{2}\right)=\sin\pi=0, z_x=z_y=\cos\pi=-1$. よって求める接平面の方程式は
$z=-\left(x-\dfrac{\pi}{2}\right)-\left(y-\dfrac{\pi}{2}\right)$. よって $x+y+z=\pi$.

5. (1) $\dfrac{dz}{dt}=\dfrac{\partial z}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial z}{\partial y}\dfrac{dy}{dt}=(\cos x\cos y)e^t+\{\sin x(-\sin y)\}\cdot\dfrac{1}{t}=e^t\cos(e^t)\cos(\log t)-\dfrac{1}{t}\sin(e^t)\sin(\log t)$.

(2) $z=\sin(e^t)\cos(\log t)$. よって $\dfrac{dz}{dt}=\{\sin(e^t)\}'\cos(\log t)+\sin(e^t)\{\cos(\log t)\}'$
$=\{\cos(e^t)\}(e^t)'\cos(\log t)+\sin(e^t)\{-\sin(\log t)\}(\log t)'=e^t\cos(e^t)\cos(\log t)-\dfrac{1}{t}\sin(e^t)\sin(\log t)$.

6. $z_u = z_x x_u + z_y y_u = \left(\dfrac{x^2}{y}\right)_x (u-2v)_u + \left(\dfrac{x^2}{y}\right)_y (2u+v)_u = \dfrac{2x}{y}\cdot 1 + x^2(-1)y^{-2}\cdot 2 = \dfrac{2x}{y} - \dfrac{2x^2}{y^2} = \dfrac{2xy-2x^2}{y^2}$
$= \dfrac{2x(y-x)}{y^2}$.
$z_v = z_x x_v + z_y y_v = \left(\dfrac{x^2}{y}\right)_x (u-2v)_v + \left(\dfrac{x^2}{y}\right)_y (2u+v)_v = \dfrac{2x}{y}\cdot(-2) + x^2(-1)y^{-2}\cdot 1 = -\dfrac{4x}{y} - \dfrac{x^2}{y^2} = -\dfrac{4xy+x^2}{y^2}$
$= -\dfrac{x(4y+x)}{y^2}$.

p.25. 練習問題 1-B

1. $x = r\cos\theta, y = r\sin\theta$ (極座標) とおくと $\dfrac{x^3+y^3}{2x^2+2y^2} = \dfrac{r^3\cos^3\theta + r^3\sin^3\theta}{2r^2\cos^2\theta + 2r^2\sin^2\theta} = \dfrac{r^3(\cos^3\theta+\sin^3\theta)}{2r^2(\cos^2\theta+\sin^2\theta)}$
$= \dfrac{r^3(\cos^3\theta+\sin^3\theta)}{2r^2} = \dfrac{r(\cos^3\theta+\sin^3\theta)}{2}$. $(x,y)\to(0,0)$ のとき $r = \sqrt{x^2+y^2}\to 0$. $-1 \leqq \sin\theta, \cos\theta \leqq 1$ より
$-2 = -1-1 \leqq \cos^3\theta + \sin^3\theta \leqq 1+1 = 2$ だから $\dfrac{r(\cos^3\theta+\sin^3\theta)}{2}\to 0$. よって $\dfrac{x^3+y^3}{2x^2+2y^2}\to 0$. 従って
$\displaystyle\lim_{(x,y)\to(0,0))} f(x,y) = \lim_{(x,y)\to(0,0))}\cos^{-1}\left(\frac{x^3+y^3}{2x^2+2y^2}\right) = \cos^{-1}0$. $\displaystyle\lim_{(x,y)\to(0,0))} f(x,y) = f(0,0) = k$ であればよいから
$k = \cos^{-1}0 = \dfrac{\pi}{2}$.

2. (1) $z_x = 2ax+by, z_y = bx+2cy$. よって $xz_x + yz_y = x(2ax+by) + y(bx+2cy) = 2ax^2+2bxy+2cy^2$
$= 2(ax^2+bxy+cy^2) = 2z$.

(2) $f(tx,ty) = t^n f(x,y)$ の左辺について $tx = X, ty = Y$ とすると $f(tx,ty) = f(X,Y)$. t で偏微分すると
$\dfrac{\partial f(tx,ty)}{\partial t} = \dfrac{\partial f(X,Y)}{\partial t} = \dfrac{\partial f(X,Y)}{\partial X}\dfrac{\partial X}{\partial t} + \dfrac{\partial f(X,Y)}{\partial Y}\dfrac{\partial Y}{\partial t} = f_X(X,Y)\cdot x + f_Y(X,Y)\cdot y$.
本来 $z = f(x,y)$ より $f_X = f_x, f_Y = f_y$ だから $\dfrac{\partial f(tx,ty)}{\partial t} = xf_x(X,Y) + yf_y(X,Y) = xf_x(tx,ty) + yf_y(tx,ty)$.
右辺を t で偏微分すると $\dfrac{\partial(t^n f(x,y))}{\partial t} = nt^{n-1}f(x,y)$ だから $xf_x(tx,ty) + yf_y(tx,ty) = nt^{n-1}f(x,y)$.
ここで $t=1$ とおくと $xf_x(x,y) + yf_y(x,y) = nf(x,y)$.

3. $\dfrac{\partial z}{\partial x} = \left(\dfrac{1}{x}\right)' f(u) + \dfrac{1}{x}(f(u))_x = -x^{-2}f(u) + \dfrac{1}{x}f'(u)u_x$. $u_x = \left(\dfrac{y}{x}\right)_x = y(-x^{-2}) = -\dfrac{y}{x^2}$ より
$\dfrac{\partial z}{\partial x} = -\dfrac{1}{x^2}f(u) - \dfrac{y}{x^3}f'(u)$. よって $x\dfrac{\partial z}{\partial x} = -\dfrac{1}{x}f(u) - \dfrac{y}{x^2}f'(u)$.
$\dfrac{\partial z}{\partial y} = \dfrac{1}{x}(f(u))_y = \dfrac{1}{x}f'(u)u_y$. $u_y = \left(\dfrac{y}{x}\right)_y = \dfrac{1}{x}$ より $\dfrac{\partial z}{\partial y} = \dfrac{1}{x^2}f'(u)$. よって $y\dfrac{\partial z}{\partial y} = \dfrac{y}{x^2}f'(u)$.
$z = \dfrac{1}{x}f(u)$ と考え合わせて $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} + z = -\dfrac{1}{x}f(u) - \dfrac{y}{x^2}f'(u) + \dfrac{y}{x^2}f'(u) + \dfrac{1}{x}f(u) = 0$.

4. $T = 2\pi\sqrt{\dfrac{l}{g}} = 2\pi l^{\frac{1}{2}}g^{-\frac{1}{2}}$ より $\dfrac{\partial T}{\partial l} = 2\pi\dfrac{1}{2}l^{-\frac{1}{2}}g^{-\frac{1}{2}} = \pi l^{-\frac{1}{2}}g^{-\frac{1}{2}}$. $\dfrac{\partial T}{\partial g} = 2\pi l^{\frac{1}{2}}\left(-\dfrac{1}{2}\right)g^{-\frac{3}{2}} = -\pi l^{\frac{1}{2}}g^{-\frac{3}{2}}$. よって
$\Delta T \fallingdotseq \dfrac{\partial T}{\partial l}\Delta l + \dfrac{\partial T}{\partial g}\Delta g = \pi l^{-\frac{1}{2}}g^{-\frac{1}{2}}\Delta l - \pi l^{\frac{1}{2}}g^{-\frac{3}{2}}\Delta g$. 両辺を $T = 2\pi l^{\frac{1}{2}}g^{-\frac{1}{2}}$ でわると
$\dfrac{\Delta T}{T} \fallingdotseq \dfrac{\pi l^{-\frac{1}{2}}g^{-\frac{1}{2}}\Delta l - \pi l^{\frac{1}{2}}g^{-\frac{3}{2}}\Delta g}{2\pi l^{\frac{1}{2}}g^{-\frac{1}{2}}} = \dfrac{\pi l^{-\frac{1}{2}}g^{-\frac{1}{2}}\Delta l}{2\pi l^{\frac{1}{2}}g^{-\frac{1}{2}}} - \dfrac{\pi l^{\frac{1}{2}}g^{-\frac{3}{2}}\Delta g}{2\pi l^{\frac{1}{2}}g^{-\frac{1}{2}}} = \dfrac{\Delta l}{2l} - \dfrac{\Delta g}{2g} = \dfrac{1}{2}\left(\dfrac{\Delta l}{l} - \dfrac{\Delta g}{g}\right)$.

5. (1) $f_x(0,y) = \displaystyle\lim_{h\to 0}\frac{f(h,y)-f(0,y)}{h} = \lim_{h\to 0}\frac{|hy|-|0\cdot y|}{h} = \lim_{h\to 0}\frac{|h||y|-0}{h} = \lim_{h\to 0}\frac{|h||y|}{h}$. $h>0$ のとき $h = |h|$ より
$\displaystyle\lim_{h\to +0}\frac{|h||y|}{h} = \lim_{h\to +0}\frac{|h||y|}{|h|} = |y|$. $h<0$ のとき $h = -|h|$ より $\displaystyle\lim_{h\to -0}\frac{|h||y|}{h} = \lim_{h\to -0}\frac{|h||y|}{-|h|} = -|y|$.
$|y| - (-|y|) = 2|y|$. $y\neq 0$ より $|y|\neq 0$ だから $|y| - (-|y|)\neq 0$. よって $|y|\neq -|y|$, 従って
$\displaystyle\lim_{h\to +0}\frac{|h||y|}{h} \neq \lim_{h\to -0}\frac{|h||y|}{h}$. よって $f_x(0,y)$ は存在しない.
同様にして $f_y(x,0) = \displaystyle\lim_{k\to 0}\frac{f(x,k)-f(x,0)}{k} = \lim_{k\to 0}\frac{|xk|-|x\cdot 0|}{k} = \lim_{k\to 0}\frac{|x||k|}{k}$. $\displaystyle\lim_{k\to +0}\frac{|x||k|}{k} = \lim_{k\to +0}\frac{|x||k|}{|k|} = |x|$,
$\displaystyle\lim_{k\to -0}\frac{|x||k|}{k} = \lim_{k\to -0}\frac{|x||k|}{-|k|} = -|x|$. $x\neq 0$ より $|x| - (-|x|) = 2|x| \neq 0$. よって $|x|\neq -|x|$,
$\displaystyle\lim_{k\to +0}\frac{|x||k|}{k} \neq \lim_{k\to -0}\frac{|x||k|}{k}$. よって $f_y(x,0)$ は存在しない.

(2) $f_x(0,0)=\lim_{h\to 0}\dfrac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\dfrac{|h\cdot 0|-|0\cdot 0|}{h}=\lim_{h\to 0}\dfrac{0-0}{h}=0.$

$f_y(0,0)=\lim_{k\to 0}\dfrac{f(0,k)-f(0,0)}{k}=\lim_{k\to 0}\dfrac{|0\cdot k|-|0\cdot 0|}{k}=\lim_{k\to 0}\dfrac{0-0}{k}=0.$

よって $\Delta z=f_x(0,0)\Delta x+f_y(0,0)\Delta y+\varepsilon$ とおくと $\Delta z=\varepsilon.$

$\Delta z=f(0+\Delta x,0+\Delta y)-f(0,0)=|\Delta x\Delta y|-|0|=|\Delta x||\Delta y|.$ より $\varepsilon=|\Delta x||\Delta y|.$ よって

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\varepsilon}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{|\Delta x||\Delta y|}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{1}{\frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{|\Delta x||\Delta y|}}$$

$$=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{1}{\sqrt{\frac{(\Delta x)^2+(\Delta y)^2}{|\Delta x|^2|\Delta y|^2}}}=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{1}{\sqrt{\frac{(\Delta x)^2}{(\Delta x)^2(\Delta y)^2}+\frac{(\Delta y)^2}{(\Delta x)^2(\Delta y)^2}}}=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{1}{\sqrt{\frac{1}{(\Delta y)^2}+\frac{1}{(\Delta x)^2}}}.$$

$\Delta x\to 0$ より $(\Delta x)^2\to +0.$ よって $\dfrac{1}{(\Delta x)^2}\to+\infty.$ 同様に $\dfrac{1}{(\Delta y)^2}\to+\infty.$ 従って $\dfrac{1}{\sqrt{\frac{1}{(\Delta y)^2}+\frac{1}{(\Delta x)^2}}}\to 0.$

よって $\lim_{(\Delta x,\Delta y)\to(0,0)}\dfrac{\varepsilon}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=0.$ 従って $f(x,y)$ は $(0,0)$ で全微分可能である.