第 3 章§ 1. 2 重積分

p.73 練習問題 1-A

1. (1) 与式$=\int_0^1\left\{\int_{2x^2}^{2x}(3y^2-xy)dy\right\}dx=\int_0^1\left[y^3-\dfrac{xy^2}{2}\right]_{2x^2}^{2x}dx=\int_0^1\{(8x^3-2x^3)-(8x^6-2x^5)\}dx$

$=\int_0^1(-8x^6+2x^5+6x^3)dx=\left[-\dfrac{8}{7}x^7+\dfrac{2}{6}x^6+\dfrac{6}{4}x^4\right]_0^1=-\dfrac{8}{7}+\dfrac{1}{3}+\dfrac{3}{2}=\underline{\dfrac{29}{42}}$

(2) 与式$=\int_1^2\left\{\int_0^1\dfrac{x}{(x+y)^2}dy\right\}dx=\int_1^2\left\{\int_0^1\{x(x+y)^{-2}dy\right\}dx=\int_1^2x\left[-(x+y)^{-1}\right]_0^1dx=\int_1^2x\{-(x+1)^{-1}+x^{-1}\}dx$

$=\int_1^2\left(-\dfrac{x}{x+1}+1\right)dx=\int_1^2\dfrac{1}{x+1}dx=[\log(x+1)]_1^2=\log 3-\log 2=\underline{\log\dfrac{3}{2}}$

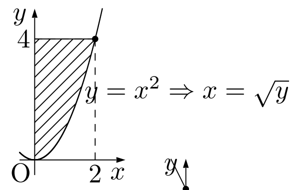
(3) 与式$=\int_0^{\frac{\pi}{2}}\left\{\int_x^{2x}\sin(2x+y)dy\right\}dx=\int_0^{\frac{\pi}{2}}[-\cos(2x+y)]_x^{2x}dx=\int_0^{\frac{\pi}{2}}(-\cos 4x+\cos 3x)dx$

$=\left[-\dfrac{\sin 4x}{4}+\dfrac{\sin 3x}{3}\right]_0^{\frac{\pi}{2}}=-\dfrac{\sin 2\pi-\sin 0}{4}+\dfrac{\sin\frac{3}{2}\pi-\sin 0}{3}=\underline{-\dfrac{1}{3}}$

(4) 与式$=\int_0^1\left\{\int_y^{\sqrt{y}}x^2dx\right\}dy=\int_0^1\left[\dfrac{x^3}{3}\right]_y^{\sqrt{y}}dx=\int_0^1\dfrac{y\sqrt{y}-y^3}{3}dy=\dfrac{1}{3}\left[\dfrac{2y^{\frac{5}{2}}}{5}-\dfrac{y^4}{4}\right]_0^1=\dfrac{1}{3}\left(\dfrac{2}{5}-\dfrac{1}{4}\right)=\underline{\dfrac{1}{20}}$

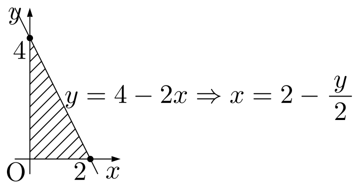
2. $D:0\leqq y\leqq 2,\ \dfrac{y}{2}\leqq x\leqq 3-y$ よって$\iint_D y^2dxdy=\int_0^2\left\{\int_{\frac{y}{2}}^{3-y}y^2dx\right\}dy=\int_0^2[y^2x]_{\frac{y}{2}}^{3-y}dy=\int_0^2y^2\left(3-y-\dfrac{y}{2}\right)dy$

$=\int_0^2\left(3y^2-\dfrac{3y^3}{2}\right)dy=\left[y^3-\dfrac{3y^4}{8}\right]_0^2=8-6=\underline{2}$



3. (1) $D:0\leqq x\leqq 2,\ x^2\leqq y\leqq 4\Rightarrow 0\leqq y\leqq 4,\ 0\leqq x\leqq\sqrt{y}$

$\int_0^2\left\{\int_{x^2}^4f(x,y)dy\right\}dx=\int_0^4\left\{\int_0^{\sqrt{y}}f(x,y)dx\right\}dy$

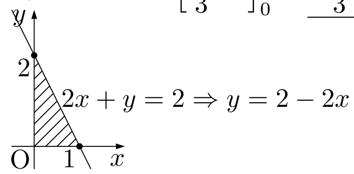


(2) $D:0\leqq y\leqq 4,\ 0\leqq x\leqq 2-\dfrac{y}{2}\Rightarrow 0\leqq x\leqq 2,\ 0\leqq y\leqq 4-2x$

$\int_0^4\left\{\int_0^{2-\frac{y}{2}}f(x,y)dx\right\}dy=\int_0^2\left\{\int_0^{4-2x}f(x,y)dy\right\}dx$

4. 曲面 $z=\sqrt{xy}$ と平面 $z=0$ の交線は $\sqrt{xy}=0$ より $x=0,y=0$ よって $D:0\leqq x\leqq 2,\ 0\leqq y\leqq 3$

$V=\iint_D\sqrt{xy}dxdy=\int_0^2\left\{\int_0^3\sqrt{xy}dy\right\}dx=\int_0^2\sqrt{x}\left[\dfrac{2}{3}y^{\frac{3}{2}}\right]_0^3dx=\int_0^2\dfrac{2}{3}\cdot 3\sqrt{3}\sqrt{x}dx=2\sqrt{3}\left[\dfrac{2}{3}x^{\frac{3}{2}}\right]_0^2=\underline{\dfrac{8\sqrt{6}}{3}}.$



5. $D:0\leqq x\leqq 1,\ 0\leqq y\leqq 2-2x(=2(1-x))$

$V=\iint_D(x^2+xy+y^2)dxdy=\int_0^1\left\{\int_0^{2(1-x)}(x^2+xy+y^2)dy\right\}dx$

$=\int_0^1\left[x^2y+\dfrac{xy^2}{2}+\dfrac{y^3}{3}\right]_0^{2(1-x)}dx=\int_0^1\left\{2x^2(1-x)+2x(1-x)^2+\dfrac{8(1-x)^3}{3}\right\}dx$

$=\int_0^1\left\{2x(1-x)\{x+(1-x)\}+\dfrac{8(1-x)^3}{3}\right\}dx=\int_0^1\left\{2x-2x^2+\dfrac{8(1-x)^3}{3}\right\}dx.$

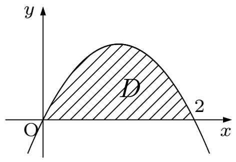
ここで $\dfrac{8(1-x)^3}{3}$ について $1-x=t$ とおくと $-dx=dt$ より $dx=-dt$

よって$\int\dfrac{8(1-x)^3}{3}dx=\int\dfrac{8t^3}{3}(-dt)=-\dfrac{2t^4}{3}=-\dfrac{2(1-x)^4}{3}$ だから

$V=\left[x^2-\dfrac{2x^3}{3}-\dfrac{2(1-x)^4}{3}\right]_0^1=\left(1-\dfrac{2}{3}-0\right)-\left(0-0-\dfrac{2}{3}\right)=\underline{1}$

p.74 練習問題 1-B

1. (1)



$y=0$ と $y=2x-x^2$ の交点は $0=2x-x^2=x(2-x)$ より $x=0,2.$

$D:0 \leqq x \leqq 2,\ 0 \leqq y \leqq 2x-x^2.$ 与式 $=\displaystyle\int_0^2 \left\{\int_0^{2x-x^2} xdy\right\}dx = \int_0^2 [xy]_0^{2x-x^2} dx$

$=\displaystyle\int_0^2 x(2x-x^2)dx = \int_0^2 (2x^2-x^3)dx = \left[\frac{2x^3}{3}-\frac{x^4}{4}\right]_0^2 = \frac{16}{3}-4=\underline{\frac{4}{3}}.$

(2) 与式 $=\displaystyle\int_0^1 \left\{\int_0^{3y} \sqrt{x+y}dx\right\}dy = \int_0^1 \left[\frac{2}{3}(x+y)^{\frac{3}{2}}\right]_0^{3y} dy = \frac{2}{3}\int_0^1 \{(4y)^{\frac{3}{2}}-y^{\frac{3}{2}}\}dy = \frac{2}{3}\int_0^1 (4^{\frac{3}{2}}-1)y^{\frac{3}{2}}dy$

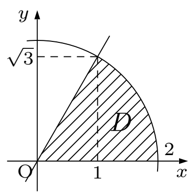
$=\displaystyle\frac{2}{3}(8-1)\left[\frac{2}{5}y^{\frac{5}{2}}\right]_0^1 = \underline{\frac{28}{15}}$

(3) 与式 $=\displaystyle\int_1^2 \left\{\int_0^{\frac{x}{2}} \frac{x}{\sqrt{x^2-y^2}}dy\right\}dx = \int_1^2 \left[x\sin^{-1}\frac{y}{x}\right]_0^{\frac{x}{2}} dx = \int_1^2 x\left(\sin^{-1}\frac{1}{2}-\sin^{-1}0\right)dx$

$\displaystyle\int_1^2 x\cdot\frac{\pi}{6}dx = \frac{\pi}{6}\left[\frac{x^2}{2}\right]_1^2 = \frac{\pi}{6}\left(2-\frac{1}{2}\right) = \underline{\frac{\pi}{4}}$

注：$\displaystyle\int \frac{1}{\sqrt{x^2-y^2}}dy = \sin^{-1}\frac{y}{x}$ は教科書 p. 177 の公式 $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}$ による.

2.



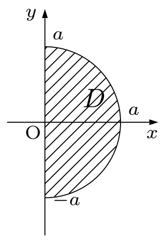
$y=\sqrt{3}x \Rightarrow x=\dfrac{x}{\sqrt{3}},\ x^2+y^2=4 \Rightarrow x=\sqrt{4-y^2}$ より

$D:0 \leqq y \leqq \sqrt{3},\ \dfrac{y}{\sqrt{3}} \leqq x \leqq \sqrt{4-y^2}$ で $\displaystyle\iint_D xdxdy = \int_0^{\sqrt{3}} \left\{\int_{\frac{y}{\sqrt{3}}}^{\sqrt{4-y^2}} xdx\right\}dy$

$=\displaystyle\int_0^{\sqrt{3}} \left[\frac{x^2}{2}\right]_{\frac{y}{\sqrt{3}}}^{\sqrt{4-y^2}} dy = \int_0^{\sqrt{3}} \left(\frac{4-y^2}{2}-\frac{y^2}{6}\right)dy = \frac{1}{6}\int_0^{\sqrt{3}} (12-4y^2)dy$

$=\displaystyle\frac{2}{3}\left[3y-\frac{y^3}{3}\right]_0^{\sqrt{3}} = \frac{2}{3}(3\sqrt{3}-\sqrt{3})\underline{\frac{4\sqrt{3}}{3}}$

3.



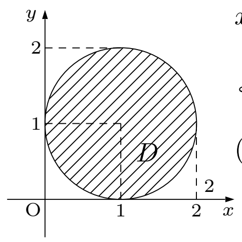
$x=\sqrt{a^2-y^2} \Rightarrow x^2+y^2=a^2 \Rightarrow y=\pm\sqrt{a^2-x^2}$ より

$D:-a \leqq y \leqq a,\ 0 \leqq x \leqq \sqrt{a^2-y^2} \Rightarrow D:x^2+y^2 \leqq a^2,\ x \geqq 0$

$\Rightarrow D:0 \leqq x \leqq a,\ -\sqrt{a^2-x^2} \leqq y \leqq \sqrt{a^2-x^2}$

よって $\displaystyle\int_{-a}^a \left\{\int_0^{\sqrt{a^2-y^2}} f(x,y)dx\right\}dy = \iint_D f(x,y)dxdy = \underline{\int_0^a \left\{\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)dy\right\}dx}$

4.



$x^2+y^2=2x+2y-1$ より $x^2-2x+y^2-2y=-1,\ (x-1)^2-1+(y-1)^2-1=-1.$

よって $(x-1)^2+(y-1)^2=1.$ $D:(x-1)^2+(y-1)^2 \leqq 1.$

$(x-1)^2+(y-1)^2=1 \Rightarrow (y-1)^2=1-(x-1)^2=2x-x^2$

$\Rightarrow y-1=\pm\sqrt{2x-x^2} \Rightarrow y=1\pm\sqrt{2x-x^2}.$

$D:0 \leqq x \leqq 2,\ 1-\sqrt{2x-x^2} \leqq y \leqq 1+\sqrt{2x-x^2}.$

$I=\displaystyle\iint_D (xy-y)dxdy = \int_0^2 \left\{\int_{1-\sqrt{2x-x^2}}^{1+\sqrt{2x-x^2}} (xy-y)dy\right\}dx = \int_0^2 \left[(x-1)\frac{y^2}{2}\right]_{1-\sqrt{2x-x^2}}^{1+\sqrt{2x-x^2}} dx$

$=\displaystyle\frac{1}{2}\int_0^2 (x-1)\{(1+\sqrt{2x-x^2})^2-(1-\sqrt{2x-x^2})^2\}dx$

$=\displaystyle\frac{1}{2}\int_0^2 (x-1)\{1+2\sqrt{2x-x^2}+2x-x^2-(1-2\sqrt{2x-x^2}+2x-x^2)\}dx$

$= \dfrac{1}{2}\displaystyle\int_0^2 (x-1)\cdot 4\sqrt{2x-x^2}dx = \int_0^2 2(x-1)\sqrt{2x-x^2}dx$

$2x-x^2=t$ とおくと $(2-2x)dx=dt$ より $-2(x-1)dx=dt,\ 2(x-1)dx=-dt.$ よって

$\displaystyle\int 2(x-1)\sqrt{2x-x^2}dy = \int \sqrt{t}(-dx) = -\frac{2}{3}t^{\frac{3}{2}} = -\frac{2}{3}(2x-x^2)^{\frac{3}{2}}.$ よって

$I = \Big[-\dfrac{2}{3}(2x-x^2)^{\frac{3}{2}}\Big]_0^2 = \underline{0}.$

5. 曲面 $z=\sqrt{a^2-y^2}$ と平面 $z=0$ の交線は $\sqrt{a^2-y^2}=0$ より $y^2=a^2$. $y \geqq \dfrac{a}{2}$ より $y=a$. よって

$D: 0 \leqq x \leqq 2a,\ \dfrac{a}{2} \leqq y \leqq a.$

$V = \displaystyle\iint_D \sqrt{a^2-y^2}dxdy = \int_0^{2a}\left\{\int_{\frac{a}{2}}^{a}\sqrt{a^2-y^2}dy\right\}dx.$

$y=a\sin\theta$ とおくと $dy=a\cos\theta d\theta,\ \sqrt{a^2-y^2}=\sqrt{a^2-a^2\sin^2\theta}=a\sqrt{1-\sin^2\theta}=a\sqrt{\cos^2\theta}=a|\cos\theta|.$

$\begin{array}{c|ccc} y & \frac{a}{2} & \to & a \\ \hline \theta & \frac{\pi}{6} & \to & \frac{\pi}{2} \end{array}$. よって $\cos\theta \geqq 0$. 半角の公式を用いて

$\displaystyle\int_{\frac{a}{2}}^{a}\sqrt{a^2-y^2}dy = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} a^2\cos^2\theta d\theta = a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\theta d\theta = a^2\left[\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = a^2\left(\frac{\pi}{4}+\frac{\sin\pi}{4}-\frac{\pi}{12}-\frac{\sin\frac{\pi}{3}}{4}\right)$

$= a^2\left(\dfrac{\pi}{6}-\dfrac{\sqrt{3}}{8}\right)$. よって

$V = a^2\left(\dfrac{\pi}{6}-\dfrac{\sqrt{3}}{8}\right)\displaystyle\int_0^{2a}dx = a^2\left(\frac{\pi}{6}-\frac{\sqrt{3}}{8}\right)[x]_0^{2a} = 2a^3\left(\frac{\pi}{6}-\frac{\sqrt{3}}{8}\right) = \underline{\frac{1}{12}a^3(4\pi-3\sqrt{3})}.$