第 3 章§ 2 変数の変換と重積分

p.92 練習問題 2-A

1. (1) $D: 0 \leqq r \leqq a, 0 \leqq \theta \leqq \frac{\pi}{2}$, 与式 $= \iint_D \sqrt{r^2} r dr d\theta = \int_0^{\frac{\pi}{2}} \left\{ \int_0^a r^2 dr \right\} d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3} \right]_0^a d\theta = \frac{a^3}{3} [\theta]_0^{\frac{\pi}{2}} = \frac{\pi a^3}{6}$

   (2) $D: 1 \leqq r \leqq 2, 0 \leqq \theta \leqq 2\pi$, 与式 $= \iint_D \frac{1}{r^2} r dr d\theta = \int_0^{2\pi} \left\{ \int_1^2 \frac{1}{r} dr \right\} d\theta = \int_0^{2\pi} [\log|r|]_1^2 d\theta = (\log 2 - \log 1)[\theta]_0^{2\pi}$
   $= 2\pi \log 2$

   (3) $D: 0 \leqq r \leqq 1, 0 \leqq \theta \leqq 2\pi$, 与式 $= \iint_D \frac{1}{1+r^2} r dr d\theta = \int_0^{2\pi} \left\{ \int_0^1 \frac{r}{1+r^2} dr \right\} d\theta = \int_0^{2\pi} \frac{1}{2} \left[\log|1+r^2| \right]_0^1 d\theta$
   $= \frac{1}{2}(\log 2 - \log 1)[\theta]_0^{2\pi} = \pi \log 2$

2. (1) (略 教科書解答参照)

   (2) $D: |u| \leqq 2, |v| \leqq 1$ よって $D: -2 \leqq u \leqq 2, -1 \leqq v \leqq 1$, $x + y = u \cdots ①, 2x - y = v \cdots ②.(① + ②)/3$ より
   $x = \frac{u+v}{3}$. $(① \times 2 - ②)/3$ より $y = \frac{2u-v}{3}$. よって $J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{3}$
   与式 $= \iint_D u^2 v^4 |J| du dv = \frac{1}{3} \int_{-2}^2 \left\{ \int_{-1}^1 u^2 v^4 dv \right\} du = \frac{4}{3} \int_0^2 u^2 \left[\frac{v^5}{5} \right]_0^1 du = \frac{4}{15} \left[\frac{u^3}{3} \right]_0^2 = \frac{32}{45}$

3. $D: x^2 + y^2 \leqq 1$, $z_x = x, z_y = y$. よって $S = \iint_D \sqrt{x^2+y^2+1} dx dy$ 極座標に変換して $D: 0 \leqq r \leqq 1, 0 \leqq \theta \leqq 2\pi$,
   $S = \iint_D \sqrt{r^2+1} r dr d\theta = \int_0^{2\pi} \left\{ \int_0^1 \sqrt{r^2+1} r dr \right\} d\theta = \int_0^{2\pi} \frac{1}{3} \left[(r^2+1)^{\frac{3}{2}} \right]_0^1 d\theta$
   $= \frac{1}{3}(2\sqrt{2} - 1)[\theta]_0^{2\pi} = \frac{2}{3}\pi(2\sqrt{2} - 1)$

4. (1) $x - 1 = t$ とおくと $dx = dt$. $\begin{array}{c|ccc} x & 1 & \to & \infty \\ \hline t & 0 & \to & \infty \end{array}$. よって p. 84 例題 4 より与式 $= \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$

   (2) $\frac{x+1}{\sqrt{2}} = t$ とおくと $dx = \sqrt{2} dt$. $\begin{array}{c|ccc} x & -\infty & \to & \infty \\ \hline t & -\infty & \to & \infty \end{array}$. よって p. 84 例題 4 と偶関数の性質より
   与式 $= \int_{-\infty}^\infty e^{-t^2} \sqrt{2} dt = 2\sqrt{2} \int_0^\infty e^{-t^2} dt = \sqrt{2\pi}$

5. 曲線 $\sqrt{x} + \sqrt{y} = 1$ は $\sqrt{y} = 1 - \sqrt{x}$ よって $y = (1 - \sqrt{x})^2$ と表わせるから問題の図形を表わす不等式は
   $D: 0 \leqq x \leqq 1, 0 \leqq y \leqq (1 - \sqrt{x})^2$ となる. $\iint_D dx dy = \int_0^1 \left\{ \int_0^{(1-\sqrt{x})^2} dy \right\} dx = \int_0^1 [y]_0^{(1-\sqrt{x})^2} dx$
   $= \int_0^1 (1 - \sqrt{x})^2 dx = \int_0^1 (1 - 2\sqrt{x} + x) dx = \left[x - \frac{4}{3} x^{\frac{3}{2}} + \frac{x^2}{2} \right]_0^1 = \frac{1}{6}$
   $\iint_D x dx dy = \int_0^1 \left\{ \int_0^{(1-\sqrt{x})^2} x dy \right\} dx = \int_0^1 x [y]_0^{(1-\sqrt{x})^2} dx = \int_0^1 x(1 - \sqrt{x})^2 dx = \int_0^1 (x - 2x\sqrt{x} + x^2) dx$
   $= \left[\frac{x^2}{2} - \frac{4}{5} x^{\frac{5}{2}} + \frac{x^3}{3} \right]_0^1 = \frac{1}{30}$. よって $\bar{x} = \frac{\frac{1}{30}}{\frac{1}{6}} = \frac{1}{5}$. 図形は直線 $y = x$ について対称だから $\bar{y} = \bar{x}$.
   よって重心は $\left(\frac{1}{5}, \frac{1}{5} \right)$

p.93 練習問題 2-B

1. $x^2 + y^2 \leqq x$ より $x^2 - x + y^2 \leqq 0$ よって $\left(x - \frac{1}{2} \right)^2 + y^2 \leqq \left(\frac{1}{2} \right)^2$. p. 77 例題 2 と同様に
   $D: 0 \leqq \theta \leqq \frac{\pi}{2}, 0 \leqq r \leqq \cos\theta$ 与式 $= 2 \iint_D \sqrt{r \cos\theta} r dr d\theta = 2 \int_0^{\frac{\pi}{2}} \left\{ \int_0^{\cos\theta} \sqrt{r \cos\theta} r dr \right\} d\theta$
   $= 2 \int_0^{\frac{\pi}{2}} \sqrt{\cos\theta} \left[\frac{2}{5} r^{\frac{5}{2}} \right]_0^{\cos\theta} d\theta = \frac{4}{5} \int_0^{\frac{\pi}{2}} \cos^3\theta d\theta = \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}$

2. p. 81 例題 3 と同様に $x = au\cos v, y = au\sin v$ とおくと $J = \begin{vmatrix} a\cos v & -au\sin v \\ b\sin v & bu\cos v \end{vmatrix} = abu$.

$D: 0 \leqq u \leqq 1, 0 \leqq v \leqq 2\pi$. 与式 $= \iint_D (a^2u^2\cos^2 v + b^2u^2\sin^2 v)abududv = ab\int_0^{2\pi}\left\{\int_0^1 u^3(a^2\cos^2 v + b^2\sin^2 v)du\right\}dv$

$= ab\int_0^{2\pi}(a^2\cos^2 v + b^2\sin^2 v)\left[\frac{u^4}{4}\right]_0^1 dv = \frac{ab}{4}\int_0^{2\pi}\left(a^2\frac{1+\cos 2v}{2} + b^2\frac{1-\cos 2v}{2}\right)dv$

$= \frac{ab}{8}\left[a^2\left(v + \frac{\sin 2v}{2}\right) + b^2\left(v - \frac{\sin 2v}{2}\right)\right]_0^{2\pi} = \frac{1}{4}\pi ab(a^2+b^2)$

3. $D_\varepsilon = \{(x,y)|\varepsilon \leqq x \leqq 1, x^2 \leqq y \leqq x\}$ とすると与式 $= \lim_{\varepsilon\to+0}\iint_{D_\varepsilon}\frac{x}{x^2+y^2}dxdy = \lim_{\varepsilon\to+0}\int_\varepsilon^1\left\{\int_{x^2}^x\frac{x}{x^2+y^2}dy\right\}dx$

p. 177 の公式 $\int\frac{dx}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$ より与式 $= \lim_{\varepsilon\to+0}\int_\varepsilon^1 x\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_{x^2}^x dx = \lim_{\varepsilon\to+0}\int_\varepsilon^1(\tan^{-1}1 - \tan^{-1}x)dx$

$= \lim_{\varepsilon\to+0}\int_\varepsilon^1\left(\frac{\pi}{4} - \tan^{-1}x\right)dx = \int_0^1\left(\frac{\pi}{4} - \tan^{-1}x\right)dx = \frac{\pi}{4} - [x\tan^{-1}x]_0^1 + \int_0^1 x\cdot\frac{1}{1+x^2}dx$

$= \frac{\pi}{4} - \tan^{-1}1 + \frac{1}{2}[\log(1+x^2)]_0^1 = \frac{\pi}{4} - \frac{\pi}{4} + \frac{1}{2}(\log 2 - \log 1) = \frac{1}{2}\log 2$

4. $x^2+y^2 = 2x$ より $x^2-2x+y^2 = 0$ よって $(x-1)^2+y^2 = 1$. p. 77 例題 2 と同様に

$D: 0 \leqq \theta \leqq \frac{\pi}{2}, 0 \leqq r \leqq 2\cos\theta$

(1) $V = 2\iint_D(x^2+y^2)dxdy = 2\iint_D r^2 rdrd\theta = 2\int_0^{\frac{\pi}{2}}\left\{\int_0^{2\cos\theta}r^3dr\right\}d\theta = 2\int_0^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^{2\cos\theta}d\theta = 8\int_0^{\frac{\pi}{2}}\cos^4\theta d\theta$

$= 8\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3}{2}\pi$

(2) $V = 2\iint_D x^2dxdy = 2\iint_D(r^2\cos^2\theta)rdrd\theta = 2\int_0^{\frac{\pi}{2}}\left\{\int_0^{2\cos\theta}r^3\cos^2\theta dr\right\}d\theta = 2\int_0^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^{2\cos\theta}\cos^2\theta d\theta$

$= 8\int_0^{\frac{\pi}{2}}\cos^6\theta d\theta = 8\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5}{4}\pi$

5. $\frac{x-\mu}{\sqrt{2}\sigma} = t$ とおくと $dx = \sqrt{2}\sigma dt, x = \sqrt{2}\sigma t + \mu$. $\begin{array}{c|ccc} x & -\infty & \to & \infty \\ \hline t & -\infty & \to & \infty \end{array}$.

よって p. 84 例題 4 と e^{-t^2} は偶関数, te^{-t^2} は奇関数だから

(1) 左辺 $= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^\infty e^{-t^2}\sqrt{2}\sigma dt = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2} = 1$

(2) 左辺 $= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^\infty(\sqrt{2}\sigma t + \mu)e^{-t^2}\sqrt{2}\sigma dt = \frac{1}{\sqrt{\pi}}\left(\sqrt{2}\sigma\int_{-\infty}^\infty te^{-t^2}dt + \mu\int_{-\infty}^\infty e^{-t^2}dt\right) = \frac{2\mu}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt$

$= \frac{2\mu}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2} = \mu$