数値計算（9）

周波数解析

（高速フーリエ変換 FFT）

離散フーリエ変換 (DFT)

$$X_k = \sum_{p=0}^{N-1} x_p \, W_N^{pk} \qquad\qquad W_N = e^{-j2\pi/N}$$

$$N = 2^m \text{ のとき,} \qquad\qquad W_N{}^m = W_N{}^{m+N}$$

$$\begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} W_4{}^0 & W_4{}^0 & W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^1 & W_4{}^2 & W_4{}^3 \\ W_4{}^0 & W_4{}^2 & W_4{}^0 & W_4{}^2 \\ W_4{}^0 & W_4{}^3 & W_4{}^2 & W_4{}^1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} W_4{}^0 & W_4{}^0 & W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^2 & W_4{}^1 & W_4{}^3 \\ W_4{}^0 & W_4{}^0 & W_4{}^2 & W_4{}^2 \\ W_4{}^0 & W_4{}^2 & W_4{}^3 & W_4{}^1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_2 \\ x_1 \\ x_3 \end{pmatrix}$$

$$= \left(\begin{array}{cc|cc} W_4{}^0 & W_4{}^0 & W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^2 & W_4{}^1 & W_4{}^3 \\ \hline W_4{}^0 & W_4{}^0 & W_4{}^2 & W_4{}^2 \\ W_4{}^0 & W_4{}^2 & W_4{}^3 & W_4{}^1 \end{array}\right) \left(\begin{array}{c} x_0 \\ x_2 \\ \hline x_1 \\ x_3 \end{array}\right) = \left(\begin{array}{cc|c} W_4{}^0 & W_4{}^0 & W_4{}^0 \times (W_4{}^0 \quad W_4{}^0) \\ W_4{}^0 & W_4{}^2 & W_4{}^1 \times (W_4{}^0 \quad W_4{}^2) \\ \hline W_4{}^0 & W_4{}^0 & W_4{}^2 \times (W_4{}^0 \quad W_4{}^0) \\ W_4{}^0 & W_4{}^2 & W_4{}^3 \times (W_4{}^0 \quad W_4{}^2) \end{array}\right) \left(\begin{array}{c} x_0 \\ x_2 \\ \hline x_1 \\ x_3 \end{array}\right)$$

$$\begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} = \left(\begin{array}{cc|c} W_4{}^0 & W_4{}^0 & W_4{}^0 \times (W_4{}^0 \quad W_4{}^0) \\ W_4{}^0 & W_4{}^2 & W_4{}^1 \times (W_4{}^0 \quad W_4{}^2) \\ \hline W_4{}^0 & W_4{}^0 & -W_4{}^0 \times (W_4{}^0 \quad W_4{}^0) \\ W_4{}^0 & W_4{}^2 & -W_4{}^1 \times (W_4{}^0 \quad W_4{}^2) \end{array}\right) \left(\begin{array}{c} x_0 \\ x_2 \\ \hline x_1 \\ x_3 \end{array}\right)$$

$$W_N{}^{m+N/2} = -W_N{}^m$$

$$W_4{}^2 = W_4{}^{0+2} = -W_4{}^0$$

$$W_4{}^3 = W_4{}^{1+2} = -W_4{}^1$$

$$\boldsymbol{A} = \begin{pmatrix} W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \boldsymbol{B}_{4,2} = \begin{pmatrix} W_4{}^0 & 0 \\ 0 & W_4{}^1 \end{pmatrix}$$

とすると

$$\begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} = \left(\begin{array}{c|c} \boldsymbol{E}_2 & \boldsymbol{B}_{4,2} \\ \hline \boldsymbol{E}_2 & -\boldsymbol{B}_{4,2} \end{array}\right) \left(\begin{array}{c} \boldsymbol{A}\begin{pmatrix} x_0 \\ x_2 \end{pmatrix} \\ \hline \boldsymbol{A}\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \end{array}\right)$$

乗算回数

$\boldsymbol{A}$は加減算のみ

$\boldsymbol{B}_{4,2}$を$\boldsymbol{A}\begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$と掛けるのに2回

$4\times4=16 \rightarrow 2$

$$
\begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \end{pmatrix}
=
\begin{pmatrix}
W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} \\
W_8^{0} & W_8^{1} & W_8^{2} & W_8^{3} & W_8^{4} & W_8^{5} & W_8^{6} & W_8^{7} \\
W_8^{0} & W_8^{2} & W_8^{4} & W_8^{6} & W_8^{0} & W_8^{2} & W_8^{4} & W_8^{6} \\
W_8^{0} & W_8^{3} & W_8^{6} & W_8^{1} & W_8^{4} & W_8^{7} & W_8^{2} & W_8^{5} \\
W_8^{0} & W_8^{4} & W_8^{0} & W_8^{4} & W_8^{0} & W_8^{4} & W_8^{0} & W_8^{4} \\
W_8^{0} & W_8^{5} & W_8^{2} & W_8^{7} & W_8^{4} & W_8^{1} & W_8^{6} & W_8^{3} \\
W_8^{0} & W_8^{6} & W_8^{4} & W_8^{2} & W_8^{0} & W_8^{6} & W_8^{4} & W_8^{2} \\
W_8^{0} & W_8^{7} & W_8^{6} & W_8^{5} & W_8^{4} & W_8^{3} & W_8^{2} & W_8^{1}
\end{pmatrix}
\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}
$$

$$
=
\left(\begin{array}{cccc|cccc}
W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} \\
W_8^{0} & W_8^{2} & W_8^{4} & W_8^{6} & W_8^{1} & W_8^{3} & W_8^{5} & W_8^{7} \\
W_8^{0} & W_8^{4} & W_8^{0} & W_8^{4} & W_8^{2} & W_8^{6} & W_8^{2} & W_8^{6} \\
W_8^{0} & W_8^{6} & W_8^{4} & W_8^{2} & W_8^{3} & W_8^{1} & W_8^{7} & W_8^{5} \\
\hline
W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{4} & W_8^{4} & W_8^{4} & W_8^{4} \\
W_8^{0} & W_8^{2} & W_8^{4} & W_8^{6} & W_8^{5} & W_8^{7} & W_8^{1} & W_8^{3} \\
W_8^{0} & W_8^{4} & W_8^{0} & W_8^{4} & W_8^{6} & W_8^{2} & W_8^{6} & W_8^{2} \\
W_8^{0} & W_8^{6} & W_8^{4} & W_8^{2} & W_8^{7} & W_8^{5} & W_8^{3} & W_8^{1}
\end{array}\right)
\left(\begin{array}{c} x_0 \\ x_2 \\ x_4 \\ x_6 \\ \hline x_1 \\ x_3 \\ x_5 \\ x_7 \end{array}\right)
$$

$$
\begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \end{pmatrix}
=
\left(\begin{array}{cccc|ccccc}
W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0\times(W_8^0 & W_8^0 & W_8^0 & W_8^0) \\
W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^1\times(W_8^0 & W_8^2 & W_8^4 & W_8^6) \\
W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^2\times(W_8^0 & W_8^4 & W_8^0 & W_8^4) \\
W_8^0 & W_8^6 & W_8^4 & W_8^2 & W_8^3\times(W_8^0 & W_8^6 & W_8^4 & W_8^2) \\
\hline
W_8^0 & W_8^0 & W_8^0 & W_8^0 & -W_8^0\times(W_8^0 & W_8^0 & W_8^0 & W_8^0) \\
W_8^0 & W_8^2 & W_8^4 & W_8^6 & -W_8^1\times(W_8^0 & W_8^2 & W_8^4 & W_8^6) \\
W_8^0 & W_8^4 & W_8^0 & W_8^4 & -W_8^2\times(W_8^0 & W_8^4 & W_8^0 & W_8^4) \\
W_8^0 & W_8^6 & W_8^4 & W_8^2 & -W_8^3\times(W_8^0 & W_8^6 & W_8^4 & W_8^2)
\end{array}\right)
\left(\begin{array}{c} x_0 \\ x_2 \\ x_4 \\ x_6 \\ \hline x_1 \\ x_3 \\ x_5 \\ x_7 \end{array}\right)
$$

$$
= \left(\begin{array}{c|c} \boldsymbol{E}_4 & \boldsymbol{B}_{8,4} \\ \hline \boldsymbol{E}_4 & -\boldsymbol{B}_{8,4} \end{array}\right)
\left(\begin{array}{c}
\begin{pmatrix}
W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
W_8^0 & W_8^4 & W_8^0 & W_8^4 \\
W_8^0 & W_8^6 & W_8^4 & W_8^2
\end{pmatrix}
\begin{pmatrix} x_0 \\ x_2 \\ x_4 \\ x_6 \end{pmatrix} \\
\hline
\begin{pmatrix}
W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
W_8^0 & W_8^4 & W_8^0 & W_8^4 \\
W_8^0 & W_8^6 & W_8^4 & W_8^2
\end{pmatrix}
\begin{pmatrix} x_1 \\ x_3 \\ x_5 \\ x_7 \end{pmatrix}
\end{array}\right)
$$

$$\begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \end{pmatrix} = \left(\begin{array}{c|c} \boldsymbol{E}_4 & \boldsymbol{B}_{8,4} \\ \hline \boldsymbol{E}_4 & -\boldsymbol{B}_{8,4} \end{array}\right) \left(\begin{array}{c} \left(\begin{array}{cc|cc} W_8{}^0 & W_8{}^0 & W_8{}^0 & W_8{}^0 \\ W_8{}^0 & W_8{}^4 & W_8{}^2 & W_8{}^6 \\ \hline W_8{}^0 & W_8{}^0 & W_8{}^4 & W_8{}^4 \\ W_8{}^0 & W_8{}^4 & W_8{}^6 & W_8{}^2 \end{array}\right) \left(\begin{array}{c} x_0 \\ x_4 \\ \hline x_2 \\ x_6 \end{array}\right) \\ \hline \left(\begin{array}{cc|cc} W_8{}^0 & W_8{}^0 & W_8{}^0 & W_8{}^0 \\ W_8{}^0 & W_8{}^4 & W_8{}^2 & W_8{}^6 \\ \hline W_8{}^0 & W_8{}^0 & W_8{}^4 & W_8{}^4 \\ W_8{}^0 & W_8{}^4 & W_8{}^6 & W_8{}^2 \end{array}\right) \left(\begin{array}{c} x_1 \\ x_5 \\ \hline x_3 \\ x_7 \end{array}\right) \end{array}\right)$$

$$= \left(\begin{array}{c|c} \boldsymbol{E}_4 & \boldsymbol{B}_{8,4} \\ \hline \boldsymbol{E}_4 & -\boldsymbol{B}_{8,4} \end{array}\right) \left(\begin{array}{c} \left(\begin{array}{c|c} \boldsymbol{E}_2 & \boldsymbol{B}_{8,2} \\ \hline \boldsymbol{E}_2 & -\boldsymbol{B}_{8,2} \end{array}\right) \left(\begin{array}{c} \boldsymbol{A} \begin{pmatrix} x_0 \\ x_4 \end{pmatrix} \\ \hline \boldsymbol{A} \begin{pmatrix} x_2 \\ x_6 \end{pmatrix} \end{array}\right) \\ \hline \left(\begin{array}{c|c} \boldsymbol{E}_2 & \boldsymbol{B}_{8,2} \\ \hline \boldsymbol{E}_2 & -\boldsymbol{B}_{8,2} \end{array}\right) \left(\begin{array}{c} \boldsymbol{A} \begin{pmatrix} x_1 \\ x_5 \end{pmatrix} \\ \hline \boldsymbol{A} \begin{pmatrix} x_3 \\ x_7 \end{pmatrix} \end{array}\right) \end{array}\right)$$

乗算回数

$8\times8=64$

↓

$2\times2+4=8$

乗算回数

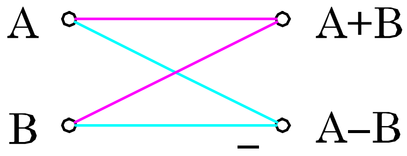
$N=2^m$ のときの，乗算回数を $M(N)$ とおくと，

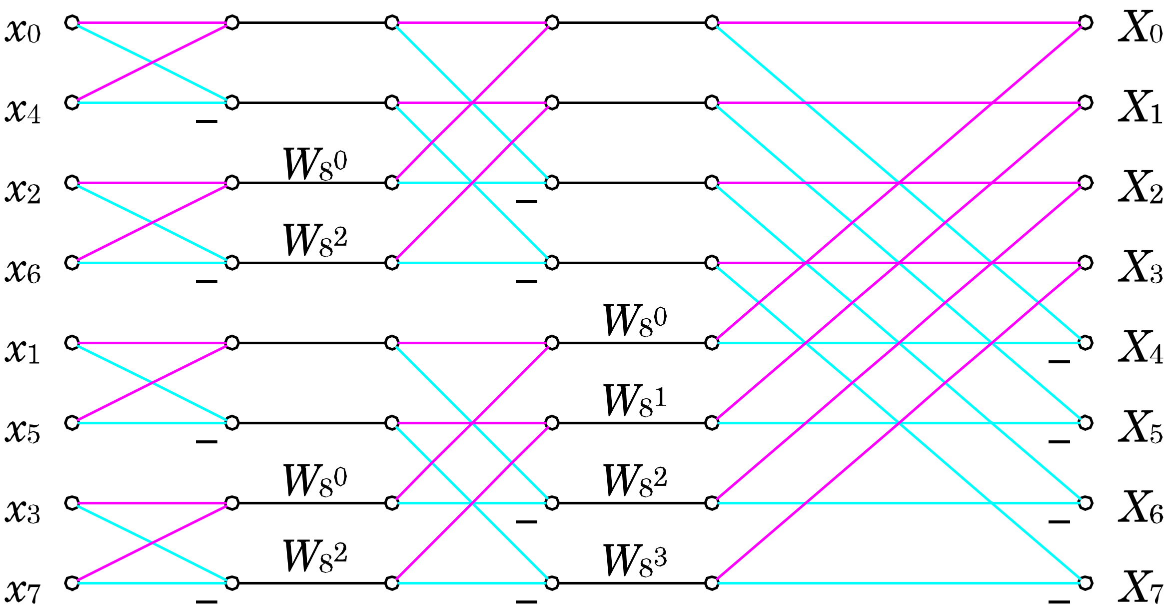
N 点の DFT を行うには，

$N/2$ の DFT を行い，$\boldsymbol{B}$ の乗算($N/2$ 回)

$$
\begin{aligned}
M(N) &= 2M(N/2) + N/2 \\
&= 2^2M(N/2^2) + 2N/2 \\
&\ \ \vdots \\
&= 2^{m-1}M(N/2^{m-1}) + (m-1)N/2 \\
&= 2^{m-1}M(2) + (m-1)N/2 \qquad M(2)=0 \\
&= (\log_2 N - 1)N/2 \\
&= (N\log_2 N - N)/2
\end{aligned}
$$

バタフライ演算





↑ 時間間引き法

並び方の変更

0	000	→	0	000
1	001	→	4	100
2	010	→	2	010
3	011	→	6	110
4	100	→	1	001
5	101	→	5	101
6	110	→	3	011
7	111	→	7	111

2進数の各桁を

逆順にした2進数順

ビット逆順

Gold-Rader 法

```
void bitreverse(data a[], int n) {
  int i, j, k;
  data w;

  j = 0;
  for (i = 0; i < n - 1; i++) {
    k = n >> 1;           /* n / 2 */
    if (i < j) {
      w = a[i];
      a[i] = a[j];
      a[j] = w;
    }
    while (k <= j) {
      j -= k;
      k >>= 1;      /* k = k / 2; */
    }
    j += k;
  }
}
```

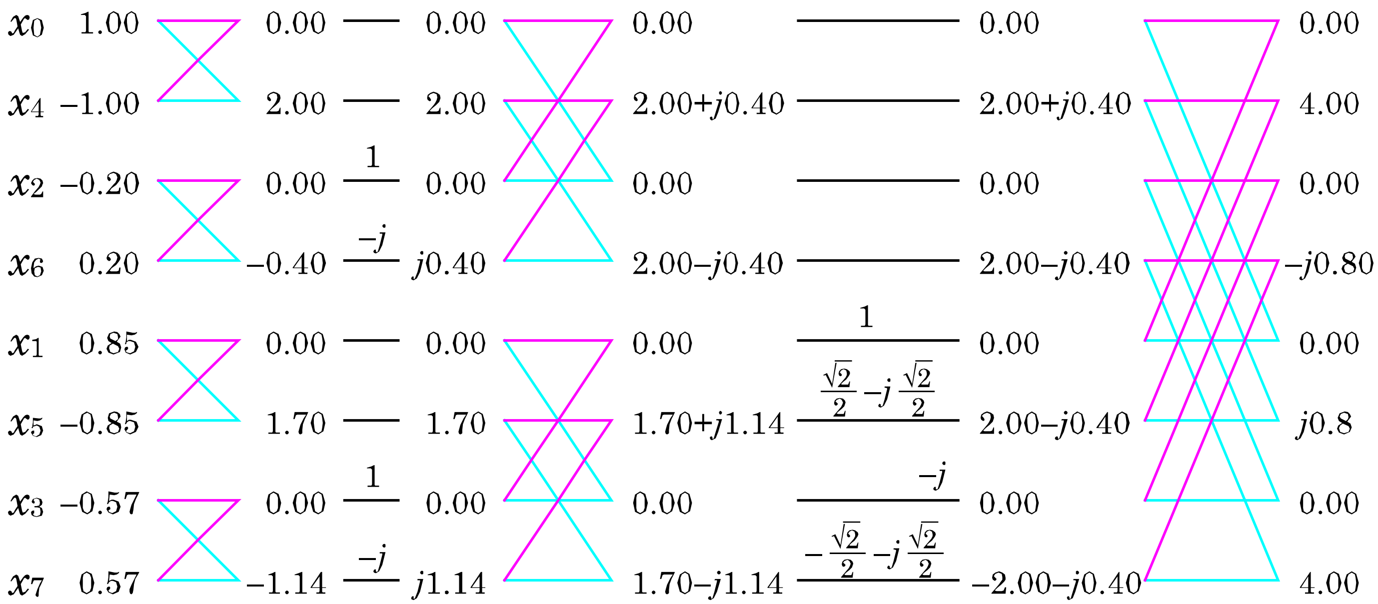
数値計算　　サンプリング間隔 = 0.5秒

x	0	1	2	3	4	5	6	7
y	1.00	0.85	−0.20	−0.57	−1.00	−0.85	0.20	0.57

$\Delta = 0.5$ [s]

$1/\Delta = 2$ [Hz]

N = 8



X_k 周波数[Hz]	X_5 −6	X_6 −4	X_7 −2	X_0 0	X_1 2	X_2 4	X_3 6	X_4 8
$Re(X_k)$	0	0	4	0	4	0	0	0
$Im(X_k)$	0.8	0	0	0	0	0	−0.8	0

周波数間引き法

$$\begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} W_4{}^0 & W_4{}^0 & W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^1 & W_4{}^2 & W_4{}^3 \\ W_4{}^0 & W_4{}^2 & W_4{}^0 & W_4{}^2 \\ W_4{}^0 & W_4{}^3 & W_4{}^2 & W_4{}^1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

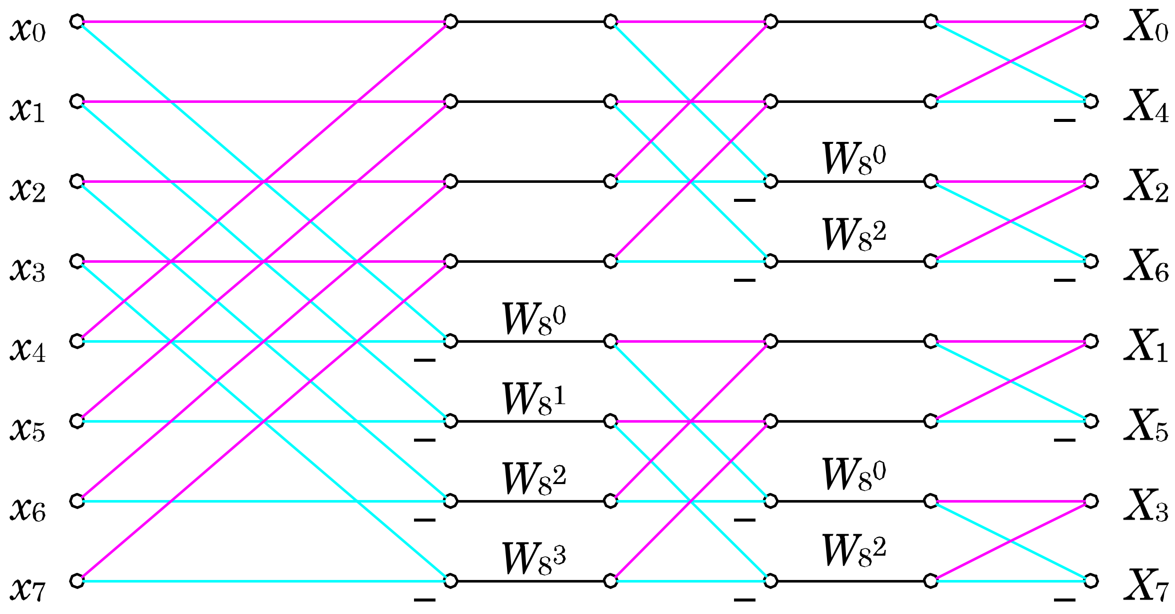
$$\begin{pmatrix} X_0 \\ X_2 \\ X_1 \\ X_3 \end{pmatrix} = \begin{pmatrix} W_4{}^0 & W_4{}^0 & W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^2 & W_4{}^0 & W_4{}^2 \\ W_4{}^0 & W_4{}^1 & W_4{}^2 & W_4{}^3 \\ W_4{}^0 & W_4{}^3 & W_4{}^2 & W_4{}^1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \left(\begin{array}{cc|cc} W_4{}^0 & W_4{}^0 & W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^2 & W_4{}^0 & W_4{}^2 \\ \hline W_4{}^0 & W_4{}^1 & W_4{}^2 & W_4{}^3 \\ W_4{}^0 & W_4{}^3 & W_4{}^2 & W_4{}^1 \end{array}\right) \left(\begin{array}{c} x_0 \\ x_1 \\ \hline x_2 \\ x_3 \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} \begin{matrix} W_4{}^0 \\ W_4{}^0 \end{matrix} & \begin{matrix} W_4{}^0 \\ W_4{}^2 \end{matrix} & \begin{matrix} W_4{}^0 \\ W_4{}^0 \end{matrix} & \begin{matrix} W_4{}^0 \\ W_4{}^2 \end{matrix} \\ \hline W_4{}^0 \times \begin{pmatrix} W_4{}^0 \\ W_4{}^0 \end{pmatrix} & W_4{}^1 \times \begin{pmatrix} W_4{}^0 \\ W_4{}^2 \end{pmatrix} & W_4{}^2 \times \begin{pmatrix} W_4{}^0 \\ W_4{}^0 \end{pmatrix} & W_4{}^3 \times \begin{pmatrix} W_4{}^0 \\ W_4{}^2 \end{pmatrix} \end{array}\right) \left(\begin{array}{c} x_0 \\ x_1 \\ \hline x_2 \\ x_3 \end{array}\right)$$

$$
\begin{pmatrix} X_0 \\ X_2 \\ X_1 \\ X_3 \end{pmatrix}
=
\left(\begin{array}{cc|cc}
W_4{}^0 & W_4{}^0 & W_4{}^0 & W_4{}^0 \\
W_4{}^0 & W_4{}^2 & W_4{}^0 & W_4{}^2 \\
\hline
W_4{}^0 \times \begin{pmatrix} W_4{}^0 \\ W_4{}^0 \end{pmatrix} & W_4{}^1 \times \begin{pmatrix} W_4{}^0 \\ W_4{}^2 \end{pmatrix} & -W_4{}^0 \times \begin{pmatrix} W_4{}^0 \\ W_4{}^0 \end{pmatrix} & -W_4{}^1 \times \begin{pmatrix} W_4{}^0 \\ W_4{}^2 \end{pmatrix}
\end{array}\right)
\left(\begin{array}{c} x_0 \\ x_1 \\ \hline x_2 \\ x_3 \end{array}\right)
$$

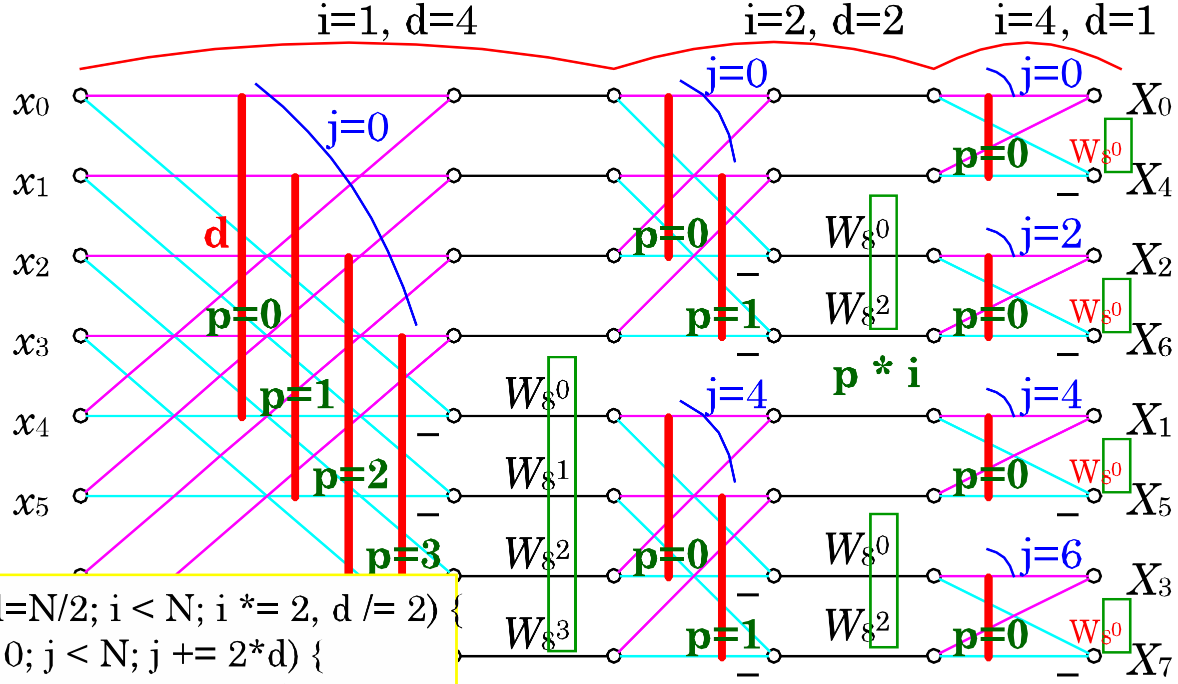
$$
\boldsymbol{A} = \begin{pmatrix} W_4{}^0 & W_4{}^0 \\ W_4{}^0 & W_4{}^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\qquad
\boldsymbol{B}_{4,2} = \begin{pmatrix} W_4{}^0 & 0 \\ 0 & W_4{}^1 \end{pmatrix}
$$

$$
\begin{pmatrix} X_0 \\ X_2 \\ X_1 \\ X_3 \end{pmatrix}
=
\left(\begin{array}{c|c}
\boldsymbol{E}_2 & \boldsymbol{E}_2 \\
\hline
\boldsymbol{B}_{4,2} & -\boldsymbol{B}_{4,2}
\end{array}\right)
\left(\begin{array}{c}
\boldsymbol{A} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \\
\hline
\boldsymbol{A} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}
\end{array}\right)
$$



x0
x1
x2
x3
x4
x5
x6
x7
W8^0
W8^1
W8^2
W8^3
W8^0
W8^2
W8^0
W8^2
X0
X4
X2
X6
X1
X5
X3
X7

周波数間引き法 ↑




```
for(i=1,d=N/2; i < N; i *= 2, d /= 2) {
  for (j = 0; j < N; j += 2*d) {
    for (p = 0; p < d; p++) {
      x_{j+p} と x_{j+p+d} でバタフライ演算
      x_{j+p+d} の方に W_N^{p*i} を掛ける
    }
  }
}
ビット逆順
```

周波数間引き法 ↑