数値計算（５）

連立１次方程式の解法（反復法）

ヤコビ法，ガウス・ザイデル法

ヤコビ法

適当な初期値を与える

$$\begin{cases} 3x + 2y + 1z = 11 \\ 1x + 4y + 2z = 12 \\ 2x + 1y + 5z = 20 \end{cases}$$

$$\begin{cases} x^{(0)} = 0 \\ y^{(0)} = 0 \\ z^{(0)} = 0 \end{cases}$$

$$\begin{cases} x = (11 - 2y - 1z) / 3 \\ y = (12 - 1x - 2z) / 4 \\ z = (20 - 2x - 1y) / 5 \end{cases}$$

$$\begin{cases} x^{(1)} = (11 - 2 \cdot 0 - 1 \cdot 0) / 3 = 3.67 \\ y^{(1)} = (12 - 1 \cdot 0 - 2 \cdot 0) / 4 = 3 \\ z^{(1)} = (20 - 2 \cdot 0 - 1 \cdot 0) / 5 = 4 \end{cases}$$

$$\begin{cases} x^{(k+1)} = (11 - 2y^{(k)} - 1z^{(k)}) / 3 \\ y^{(k+1)} = (12 - 1x^{(k)} - 2z^{(k)}) / 4 \\ z^{(k+1)} = (20 - 2x^{(k)} - 1y^{(k)}) / 5 \end{cases}$$

$$\begin{cases} x^{(2)} = (11 - 2 \cdot 3 - 1 \cdot 4) / 3 = 0.33 \\ y^{(2)} = (12 - 1 \cdot 3.67 - 2 \cdot 4) / 4 = 0.08 \\ z^{(2)} = (20 - 2 \cdot 3.67 - 1 \cdot 3) / 5 = 1.93 \end{cases}$$

$$\begin{cases} x^{(3)} = (11 - 2 \cdot 0.08 - 1 \cdot 1.93) \,/\, 3 = 2.97 \\ y^{(3)} = (12 - 1 \cdot 0.33 - 2 \cdot 1.93) \,/\, 4 = 1.95 \\ z^{(3)} = (20 - 2 \cdot 0.33 - 1 \cdot 0.08) \,/\, 5 = 3.85 \end{cases}$$

$$\vdots$$

$$\begin{cases} x^{(20)} = (11 - 2 \cdot 1.02 - 1 \cdot 3.02) \,/\, 3 = 1.98 \\ y^{(20)} = (12 - 1 \cdot 2.02 - 2 \cdot 3.02) \,/\, 4 = 0.99 \\ z^{(20)} = (20 - 2 \cdot 2.02 - 1 \cdot 0.02) \,/\, 5 = 2.99 \end{cases}$$

収束判定

$$x^{(20)} - x^{(19)} = -0.04$$

$$y^{(20)} - y^{(19)} = 0.03$$

$$z^{(20)} - z^{(19)} = 0.03$$

変化は十分少ない

一般に

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

$$A\boldsymbol{x} = \boldsymbol{b}$$

①
$$\begin{cases} x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - \cdots - a_{1n}x_n) \\ x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - \cdots - a_{2n}x_n) \\ \vdots \\ x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots \quad) \end{cases}$$

$$D = \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix}$$

とすると

$$D\boldsymbol{x} = \boldsymbol{b} - (A - D)\boldsymbol{x}$$

$$\boldsymbol{x} = D^{-1}(\boldsymbol{b} - (A - D)\boldsymbol{x})$$

反復式

$$\begin{cases} x_1^{(k+1)} = \dfrac{1}{a_{11}}(b_1 - a_{12}x_2^{(k)} - \cdots - a_{1n}x_n^{(k)}) \\ x_2^{(k+1)} = \dfrac{1}{a_{22}}(b_2 - a_{21}x_1^{(k)} - \cdots - a_{2n}x_n^{(k)}) \\ \quad\vdots \\ x_n^{(k+1)} = \dfrac{1}{a_{nn}}(b_n - a_{n1}x_1^{(k)} - a_{n2}x_2^{(k)} - \cdots) \end{cases}$$

$$\boldsymbol{x}^{(k+1)} = D^{-1}(\boldsymbol{b} - (A - D)\boldsymbol{x}^{(k)})$$

収束判定条件

$$\sqrt{\sum_{i=1}^{n}(x_i^{(k+1)} - x_i^{(k)})^2} < \varepsilon \ (\text{十分小さい数}) \qquad \left|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\right| < \varepsilon$$

他に

$$|x_i^{(k+1)} - x_i^{(k)}| < \varepsilon \quad (1 \leqq i \leqq n)$$

$$|x_i^{(k+1)} - x_i^{(k)}| < \varepsilon |x_i^{(k)}| \quad (1 \leqq i \leqq n)$$

など

収束条件

①－反復式　→　誤差

$$x_1 - x_1^{(k+1)} = -\frac{1}{a_{11}}\{a_{12}(x_2 - x_2^{(k)}) - \cdots - a_{1n}(x_n - x_n^{(k)})\}$$

$$x_2 - x_2^{(k+1)} = -\frac{1}{a_{22}}\{a_{21}(x_1 - x_1^{(k)}) - \cdots - a_{2n}(x_n - x_n^{(k)})\}$$

$$\vdots$$

$$x_n - x_n^{(k+1)} = -\frac{1}{a_{nn}}\{a_{n1}(x_1 - x_1^{(k)}) - a_{n2}(x_2 - x_2^{(k)}) - \cdots \}$$

$$(\boldsymbol{x} - \boldsymbol{x}^{(k+1)}) = D^{-1}(A - D)(\boldsymbol{x} - \boldsymbol{x}^{(k)})$$

$D^{-1}(A-D)$ の固有値がすべて1以下なら必ず収束

固有値 λ , 固有ベクトル $\boldsymbol{y}$ とすると,

$$\lambda y_i = -\frac{1}{a_{ii}}(a_{i1}y_1 + \cdots 0 \cdot y_i \cdots + a_{in}y_n) \qquad (1 \leqq i \leqq n)$$

$$= -\sum_{i \neq j} \frac{a_{ij}}{a_{ii}} y_j \qquad (1 \leqq i \leqq n)$$

$$\lambda y_i = -\sum_{i \neq j} \frac{a_{ij}}{a_{ii}} y_j \qquad (1 \leqq i \leqq n)$$

両辺絶対値をとり $|y_i|$ で割ると，

$$|\lambda| \leqq \sum_{i \neq j} \frac{|a_{ij}|}{|a_{ii}|} \frac{|y_j|}{|y_i|} \qquad (1 \leqq i \leqq n)$$

$|y_i| = \max\{|y_1|, \ldots, |y_n|\}$ のとき， $\dfrac{|y_j|}{|y_i|} \leqq 1$ より，

$$|\lambda| \leqq \sum_{i \neq j} \frac{|a_{ij}|}{|a_{ii}|}$$

よって， $\displaystyle\sum_{i \neq j} \frac{|a_{ij}|}{|a_{ii}|} < 1$ すなわち

$$|a_{ii}| > \sum_{i \neq j} |a_{ij}| \quad \text{なら，必ず収束}$$

（十分条件）

ガウス・ザイデル法

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$$\begin{cases} x = (11 - 2y - 1z) / 3 \\ y = (12 - 1x - 2z) / 4 \\ z = (20 - 2x - 1y) / 5 \end{cases}$$

$$\begin{cases} x^{(k+1)} = (11 - 2y^{(k)} - 1z^{(k)}) / 3 \\ y^{(k+1)} = (12 - 1x^{(k+1)} - 2z^{(k)}) / 4 \\ z^{(k+1)} = (20 - 2x^{(k+1)} - 1y^{(k+1)}) / 5 \end{cases}$$

適当な初期値を与える

$$\begin{cases} x^{(0)} = 0 \\ y^{(0)} = 0 \\ z^{(0)} = 0 \end{cases}$$

$$\begin{cases} x^{(1)} = (11 - 2 \cdot 0 - 1 \cdot 0) / 3 = 3.67 \\ y^{(1)} = (12 - 1 \cdot 3.67 - 2 \cdot 0) / 4 = 2.08 \\ z^{(1)} = (20 - 2 \cdot 3.67 - 1 \cdot 2.08) / 5 = 2.12 \end{cases}$$

$$\begin{cases} x^{(2)} = (11 - 2 \cdot 2.08 - 1 \cdot 2.12) / 3 = 1.57 \\ y^{(2)} = (12 - 1 \cdot 1.57 - 2 \cdot 2.12) / 4 = 1.55 \\ z^{(2)} = (20 - 2 \cdot 1.57 - 1 \cdot 1.55) / 5 = 3.06 \end{cases}$$

$$\begin{cases} x^{(3)} = (11 - 2\cdot1.55 - 1\cdot3.06) / 3 = 1.61 \\ y^{(3)} = (12 - 1\cdot1.61 - 2\cdot3.06) / 4 = 1.07 \\ z^{(3)} = (20 - 2\cdot1.61 - 1\cdot1.07) / 5 = 3.14 \end{cases}$$

$$\begin{cases} x^{(4)} = (11 - 2\cdot1.07 - 1\cdot3.14) / 3 = 1.91 \\ y^{(4)} = (12 - 1\cdot1.91 - 2\cdot3.14) / 4 = 0.95 \\ z^{(4)} = (20 - 2\cdot1.91 - 1\cdot0.95) / 5 = 3.05 \end{cases}$$

$$\begin{cases} x^{(5)} = (11 - 2\cdot0.95 - 1\cdot3.05) / 3 = 2.02 \\ y^{(5)} = (12 - 1\cdot2.02 - 2\cdot3.05) / 4 = 0.97 \\ z^{(5)} = (20 - 2\cdot2.02 - 1\cdot0.97) / 5 = 3.00 \end{cases}$$

一般に

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

$$A\boldsymbol{x} = \boldsymbol{b}$$

$$\begin{cases} x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - \cdots - a_{1n}x_n) \\ x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - \cdots - a_{2n}x_n) \\ \vdots \\ x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots) \end{cases}$$

$$E = \begin{pmatrix} 0 & & & 0 \\ a_{21} & \ddots & & \\ \vdots & & \ddots & \\ a_{n2} & & & 0 \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & a_{21} & \cdots & a_{1n} \\ & \ddots & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

$$\boldsymbol{x} = D^{-1}(\boldsymbol{b} - E\boldsymbol{x} - F\boldsymbol{x})$$

反復式

$$\begin{cases} x_1^{(k+1)} = \dfrac{1}{a_{11}}(b_1 - a_{12}x_2^{(k)} - \cdots - a_{1n}x_n^{(k)}) \\ x_2^{(k+1)} = \dfrac{1}{a_{22}}(b_2 - a_{21}x_1^{(k+1)} - \cdots - a_{2n}x_n^{(k)}) \\ \quad\vdots \\ x_n^{(k+1)} = \dfrac{1}{a_{nn}}(b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} - \cdots) \end{cases}$$

$$\boldsymbol{x}^{(k+1)} = D^{-1}(\boldsymbol{b} - E\boldsymbol{x}^{(k+1)} - F\boldsymbol{x}^{(k)})$$

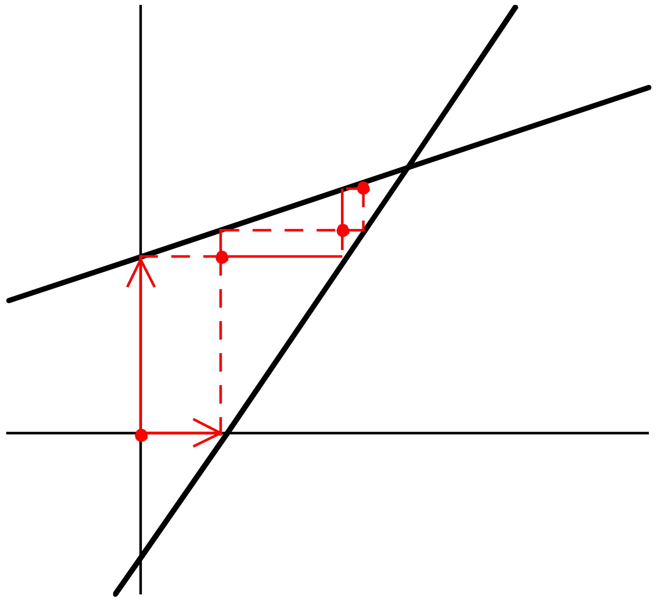
収束判定条件はヤコビ法と同様

収束条件もヤコビ法と同様に

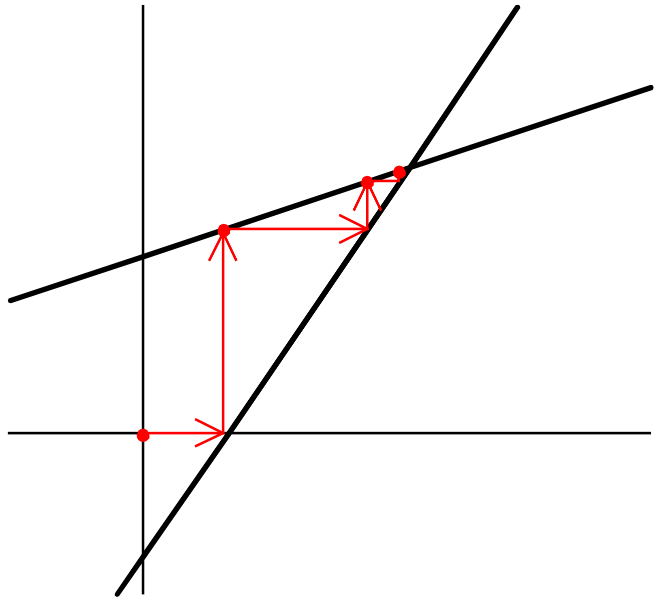
$$|a_{ii}| > \sum_{i \neq j} |a_{ij}|$$

（十分条件）

幾何学的意味



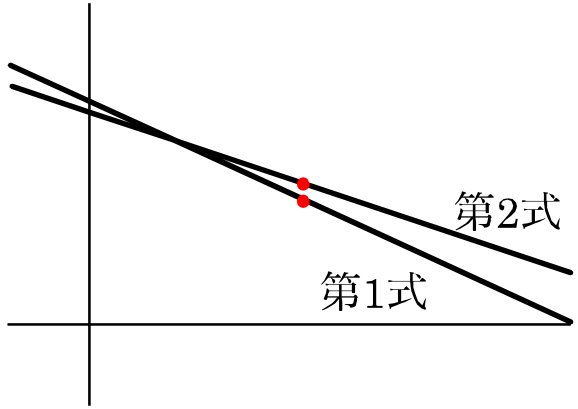
ヤコビ法



ガウス・ザイデル法

収束判定

$$\begin{cases} 0.4x_1 + \ \ 1.2x_2 \ \ = \ \ 5.2 \\ 3.5x_1 + 10.501x_2 = 45.504 \end{cases}$$



解析解　$(x_1, x_2) = (1, 4)$

仮に，$\begin{cases} x_1{}^{(k)} = 10 \\ x_2{}^{(k)} = 1 \end{cases}$　になったとすると

$$\begin{cases} x_1{}^{(k+1)} = (5.2 - 1.2\cdot 1) \,/\, 0.4 = 10 \\ x_2{}^{(k+1)} = (45.504 - 3.5\cdot 10) \,/\, 10.501 = 1.002856\cdots \end{cases}$$

$$\sqrt{\sum_{i=1}^{n}(x_i{}^{(k+1)} - x_i{}^{(k)})^2} = \sqrt{(10-10)^2 + (1.00286-1)^2} = 0.00286$$