α の n 乗根

[p.123—125]

n 次方程式 $w^n=\alpha$ を満たす解 w のことを複素数 α の n乗根 (n-th root) という．このwの値は n 個ある．また，記号 $\alpha^{\frac{1}{n}}$ で表す．

5a 複素数 α の n 乗根を求めよ． (一般的な表式)

[p.124]

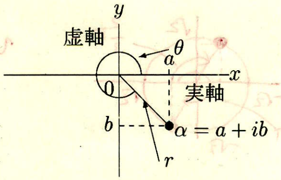
[解] n 次方程式 $w^n=\alpha$ を解く．$w=\rho e^{i\varphi}$ とおく．
また右図のように $\alpha = r(\cos\theta+i\sin\theta)=re^{i\theta}$ とする．

$\therefore (\rho e^{i\varphi})^n = re^{i\theta}, \qquad \therefore \rho^n (e^{i\varphi})^n = re^{i\theta},$

ド・モアブルの定理より，$\rho^n e^{in\varphi} = re^{i\theta}$.

$$\therefore \begin{cases} \rho^n = r \\ n\varphi = \theta + 2k\pi \end{cases} \qquad (k=0,\pm1,\pm2,\cdots).$$

$$\therefore \begin{cases} \rho = r^{\frac{1}{n}} = \sqrt[n]{r} \\ \varphi = \dfrac{\theta}{n} + \dfrac{2k\pi}{n} \end{cases} \qquad (k=\overbrace{0,1,2,\cdots,n-1}^{n\text{ 個}})$$



y
虚軸
θ
a
x
0
実軸
b
α = a + ib
r

図：α の絶対値と偏角 (極形式)

$$\therefore w\left(=\alpha^{\frac{1}{n}}\right) = \sqrt[n]{r}e^{i\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)} = \sqrt[n]{r}\exp\left(i\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right) \quad /\!/$$

$$\therefore w\left(=\alpha^{\frac{1}{n}}\right) = \sqrt[n]{r}\left\{\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right\}, /\!/ \quad (k=\overbrace{0,1,2,\cdots,n-1}^{n\text{ 個}})$$

$\boxed{注}$：$\alpha^{\frac{1}{n}}$ は n 個ある．実数 $r\,(r\geq 0)$ の n 乗根 $r^{\frac{1}{n}}=\sqrt[n]{r}$ は1個のみと考える．

関連問題 [p.125 1.][p.131 4.]

5b (1) 複素数 $\sqrt{2}+\sqrt{6}i$ を複素平面上に描き，極形式を求めよ．
(2) $\sqrt{2}+\sqrt{6}i$ の4乗根 w が満たす方程式を書き，$w=\rho e^{i\phi}$ とおき，その方程式を極形式に直せ．
(3) w の絶対値と偏角を求め，w を極形式ですべて求めよ．
(4) w を複素平面上に描き，通常の $a+ib$ の形で表せる数はその形に直せ．

(1)

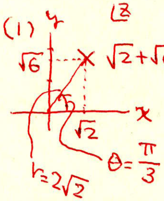


図
y
√6
x
√2
θ = π/3
r = 2√2

$\sqrt{2}+\sqrt{6}\,i = 2\sqrt{2}\,e^{i\frac{\pi}{3}}$

(2) $w^4 = \sqrt{2}+\sqrt{6}\,i$

$(\rho e^{i\varphi})^4 = 2\sqrt{2}\,e^{i\frac{\pi}{3}}, \quad \rho^4 e^{i4\varphi} = 2\sqrt{2}\,e^{i\frac{\pi}{3}}$

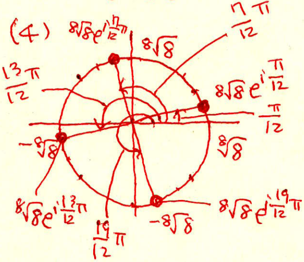
(3)
$$\begin{cases} \rho^4 = 2\sqrt{2} = \sqrt{8} = 8^{\frac{1}{2}} \\ 4\varphi = \dfrac{\pi}{3}+2k\pi \quad (k=0,\pm1,\pm2,\cdots) \end{cases}$$

$$\begin{cases} \rho = 8^{\frac{1}{8}} = \sqrt[8]{8} \\ \varphi = \dfrac{\pi}{12}+\dfrac{k\pi}{2} \quad (k=\overbrace{0,1,2,3}^{4個}) \end{cases}$$

$= \dfrac{\pi}{12}, \dfrac{7}{12}\pi, \dfrac{13}{12}\pi, \dfrac{19}{12}\pi$

$$w = \begin{cases} \sqrt[8]{8}\,e^{i\frac{\pi}{12}} \\ \sqrt[8]{8}\,e^{i\frac{7}{12}\pi} \\ \sqrt[8]{8}\,e^{i\frac{13}{12}\pi} \\ \sqrt[8]{8}\,e^{i\frac{19}{12}\pi} \end{cases}$$

(4)



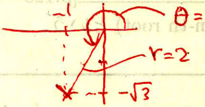
$\sqrt[8]{8}e^{i\frac{7}{12}\pi}$
$\sqrt[8]{8}$
$\frac{7}{12}\pi$
$\frac{13}{12}\pi$
$\sqrt[8]{8}e^{i\frac{\pi}{12}}$
$\frac{\pi}{12}$
$-\sqrt[8]{8}$
$\sqrt[8]{8}$
$\sqrt[8]{8}e^{i\frac{13}{12}\pi}$
$\frac{19}{12}\pi$
$-\sqrt[8]{8}$
$\sqrt[8]{8}e^{i\frac{19}{12}\pi}$

中心O半径$\sqrt[8]{8}$の円を
4等分した点．

関連問題 [p.125 1.][p.131 4.]

5c 前問 **5b** と同様の手順で, $-1-\sqrt{3}i$ の平方根, $-8i$ の 3 乗根, $1-i$ の 3 乗根をそれぞれ求めよ.

(1)



$\theta = \pi + \frac{\pi}{3} = \frac{4}{3}\pi$

$-1-\sqrt{3}i = 2e^{i\frac{4}{3}\pi}$

(2) $w^2 = -1-\sqrt{3}i$

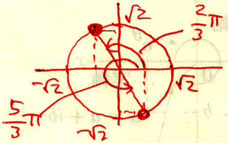
$(\rho e^{i\varphi})^2 = 2e^{i\frac{4}{3}\pi}$ $\quad \rho^2 e^{i2\varphi} = 2e^{i\frac{4}{3}\pi}$

(3) $\begin{cases} \rho^2 = 2 \\ 2\varphi = \frac{4}{3}\pi + 2k\pi \end{cases}$ $\quad \rho = \sqrt{2}$, $\varphi = \frac{2}{3}\pi + k\pi$ $(\underbrace{k=0,1}_{}$ 2個$)$

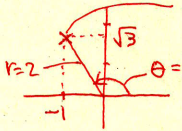
$= \frac{2}{3}\pi, \frac{5}{3}\pi$

$w = \begin{cases} \sqrt{2}e^{i\frac{2}{3}\pi} = \sqrt{2}\left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right) = \sqrt{2}\,\frac{-1+\sqrt{3}i}{2} \\ \sqrt{2}e^{i\frac{5}{3}\pi} = \sqrt{2}\left(\cos\frac{5}{3}\pi + i\sin\frac{5}{3}\pi\right) = \sqrt{2}\,\frac{1-\sqrt{3}i}{2} \end{cases}$ //

(4)


$\sqrt{2}$
$\frac{2}{3}\pi$
$-\sqrt{2}$
$\sqrt{2}$
$\frac{5}{3}\pi$
$-\sqrt{2}$

$-1+\sqrt{3}i$ の 3 乗根



$-1+\sqrt{3}i = 2e^{i\frac{2}{3}\pi}$

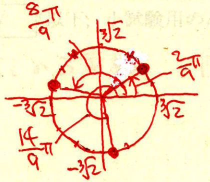
$\theta = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2}{3}\pi$

$w^3 = -1+\sqrt{3}i$

$(\rho e^{i\varphi})^3 = 2e^{i\frac{2}{3}\pi}$ $\quad \rho^3 e^{i3\varphi} = 2e^{i\frac{2}{3}\pi}$

$\begin{cases} \rho^3 = 2 \\ 3\varphi = \frac{2}{3}\pi + 2k\pi \end{cases}$ $\quad \begin{cases} \rho = \sqrt[3]{2} \\ \varphi = \frac{2}{9}\pi + \frac{2k\pi}{3} \quad (k=0,1,2) \end{cases}$

$= \frac{2}{9}\pi, \frac{8}{9}\pi, \frac{14}{9}\pi$


$\frac{8}{9}\pi$
$\sqrt[3]{2}$
$\frac{2}{9}\pi$
$-\sqrt[3]{2}$
$\sqrt[3]{2}$
$\frac{14}{9}\pi$
$-\sqrt[3]{2}$

$w = \begin{cases} \sqrt[3]{2}e^{i\frac{2}{9}\pi} \\ \sqrt[3]{2}e^{i\frac{8}{9}\pi} \\ \sqrt[3]{2}e^{i\frac{14}{9}\pi} \end{cases}$