複素関数

実数から$\boxed{\text{複素数}}$に拡張することにより，単純で明解な枠組が数学理論にあることが理解できる．また，$\boxed{\text{複素平面}}$，$\boxed{\text{複素変数}}$，$\boxed{\text{複素関数}}$に拡張することにより，単純で明解な数学的表現や複素$\boxed{\text{関数の微積分}}$などの理論があることが分かり，読解・表現・計算できる．

複素数 (complex number) の定義　　複素数とは？

[p.116]

$\alpha = a + ib = a + bi$　　　$\alpha = Re(\alpha) + iIm(\alpha)$

$i = \sqrt{-1}$　虚数単位 (imaginary unit)　　$i^2 = -1$

$a = Re(\alpha)$　実部 (real part)

$b = Im(\alpha)$　虚部 (imaginary part)

a, b は　実数　(real number)

[p.116]

$\alpha = a = a + i0$ は　実数　(real number)　　$Im(\alpha) = 0$

$\alpha = ib = 0 + ib$ は　純虚数　(pure imaginary number)　　$Re(\alpha) = 0$

$0 = 0 + i0$

[p.116]

$\alpha = a + ib, \quad \beta = c + id$ の計算

相等 $\alpha = \beta \Leftrightarrow \begin{cases} a = c & \text{実部} \\ b = d & \text{虚部} \end{cases}$　とくに，$\alpha = 0 \Leftrightarrow \begin{cases} a = 0 \\ b = 0 \end{cases}$

四則演算

$$\alpha + \beta = (a + ib) + (c + id) = a + ib + c + id = (a+c) + i(b+d)$$

$$\alpha - \beta = (a + ib) - (c + id) = a + ib - c - id = (a-c) + i(b-d)$$

$$\alpha\beta = (a + ib)(c + id) = ac + iad + ibc + i^2bd = (ac-bd) + i(ad+bc)$$

$$\frac{\alpha}{\beta} = \frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{ac - iad + ibc - i^2bd}{c^2 - i^2d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

代数計算をし，$i^2 = -1$ を代入．実部と虚部に整理して，結果も複素数．

[p.116—117]

$Re(\alpha\beta) = ac - bd$　　　$Im\left(\dfrac{\alpha}{\beta}\right) = \dfrac{bc-ad}{c^2+d^2}$

共役複素数 (complex conjugate) $\overline{\alpha}$　　共役複素数とは？

[p.116]

$\alpha = a + ib$ に対して　　$\overline{\alpha} = a - ib$　虚部の符号を変える

[p.117—118]

$\overline{\overline{\alpha}} = \alpha$

$\alpha + \overline{\alpha} = (a+\cancel{ib}) + (a-\cancel{ib}) = 2a = 2Re(\alpha)$　　$Re(\alpha) = \dfrac{\alpha+\overline{\alpha}}{2}$

$\alpha - \overline{\alpha} = (\cancel{a}+ib) - (\cancel{a}-ib) = 2ib = 2iIm(\alpha)$　　$Im(\alpha) = \dfrac{\alpha-\overline{\alpha}}{2i}$

[p.117]

$\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$

$\overline{\alpha - \beta} = \overline{\alpha} - \overline{\beta}$

$\overline{\alpha\beta} = \overline{\alpha}\,\overline{\beta}$

$\overline{\left(\dfrac{\alpha}{\beta}\right)} = \dfrac{\overline{\alpha}}{\overline{\beta}}$

[p.118]

例 1 $\dfrac{4-2i}{3+i} = \dfrac{(4-2i)(3-i)}{(3+i)(3-i)} = \dfrac{12-4i-6i+2i^2}{9-i^2} = \dfrac{10-10i}{10} = 1-i \quad \|$

[p.117 例 1]

関連問題 [p.117 問 1][p.131 1.]

1a 次の計算をせよ. (a) $(-2+7i)-(8-6i) = -2+7i-8+6i = -10+13i$

(b) $(5-2i)(3+2i) = 15+10i-6i-4i^2 = 19+4i$

(c) $\dfrac{3+i}{1+i} = \dfrac{(3+i)(1-i)}{(1+i)(1-i)} = \dfrac{3-3i+i-i^2}{1^2-i^2} = \dfrac{4}{2}+\dfrac{-2i}{2} = 2-i$

(d) $i^3 = i^2 \cdot i = -i$ $\qquad i^4 = i^2 \cdot i^2 = 1$ $\qquad i^5 = i^2 \cdot i^2 \cdot i = i$

(e) $\dfrac{2i^5}{1+i^3} = \dfrac{2i}{1-i} = \dfrac{2i(1+i)}{(1-i)(1+i)} = \dfrac{2i+2i^2}{1^2-i^2} = \dfrac{-2}{2}+\dfrac{2i}{2} = -1+i$

(f) $Re\left(\dfrac{3+i}{1+i}\right) = Re(2-i) = 2$ $\qquad$ (g) $Im\left(\dfrac{2i^5}{1+i^3}\right) = Im(-1+i) = 1$

関連問題 [p.117 問 2][p.118 問 3 問 4]

1b $\alpha = a+ib$, $\beta = c+id$ として,

(a) $\overline{\alpha\beta} = \overline{\alpha}\,\overline{\beta}$ $\quad \because \overline{\alpha\beta} = \overline{(a+ib)(c+id)} = \overline{ac+aid+ibc+i^2bd}$

$= \overline{(ac-bd)+i(ad+bc)} = (ac-bd)-i(ad+bc)$

$\overline{\alpha}\,\overline{\beta} = (a-ib)(c-id) = ac-aid-ibc+i^2bd = (ac-bd)-i(ad+bc)$

(b) $\alpha\overline{\beta}-\overline{\alpha}\beta = (a+ib)(c-id)-(a-ib)(c+id)$

$= (\cancel{ac}-aid+ibc-\cancel{i^2bd})-(\cancel{ac}+aid-ibc-\cancel{i^2bd}) = 2i(bc-ad)$ 純虚数

$(5+2i)(3-2i) = 15-10i+6i-4i^2 = 19-4i$

$\dfrac{1+i}{3+i} = \dfrac{(1+i)(3-i)}{(3+i)(3-i)} = \dfrac{3-i+3i-i^2}{3^2-i^2} = \dfrac{4}{10}+\dfrac{+2i}{10} = \dfrac{2}{5}+\dfrac{i}{5}$

$Re\left((5+2i)(3-2i)\right) = Re(19-4i) = 19$

$Im\left(\dfrac{1+i}{3+i}\right) = Im\left(\dfrac{2}{5}-\dfrac{i}{5}\right) = +\dfrac{1}{5}$